PROXIMITY SPACES WITH APPLICATIONS

TO COMPACTIFICATIONS

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B.S. Kansas State University, 1968

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

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Manhattan, Kansas

1969

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Introduction

Our main concern is problems in and related to compactifications of completely regular spaces. However, we find that,in general, compactifications are characterized "externally." For example, see the characterization of the Stone-Čech compactification in ([1], XI, Theorem 8.2 1)). It would be profitable to find some "internal" characterization, or at least a (simple) characterization in terms of the topology on the space to be compactified.

We proceed to find a "representation" of compactifications, and therefore of their problems, in a setting where the problems are, hopefully, more accessible. For this purpose we chose the class of proximity spaces, a strong motivation for this choice being the existence of an isomorphism between the set of compactifications of a given completely regular space and the set of proximity relations consistent with the space (see theorem 4.5).

In chapter I of this paper, we present some of the topological prerequisites for a detailed study of proximity spaces and their application to other aspects of mathematics, in particular, to compactifications of topological spaces. After which, in chapter II, we define and prove some basic properties of proximity spaces. We then go to chapter III and prove that the category of proximity spaces coincides exactly with the category of completely regular spaces. This enables us in chapter IV to establish a 1-1 correspondence between the compactifications of a completely regular space and the proximity relations defined on that space. In fact, we establish an isomorphism between the compactifications of, and the proximity relations associated with, a given completely regular space (cf. 4.5). We then apply the theory of proximity spaces to the study of compactifications of completely regular spaces. We submit that the use of the theory of proximity spaces in chapter V -- to prove an extension theorem for homeomorphisms, and an important result of Freudenthal-Morita on the existence of a compactification, with zero-dimensional annex, for every peripherally compact space -- demonstrates the usefulness of the theory. The result in chapter V, in addition to their intrinsic value, give rise to renewed hope of solving some old problems in the theory of compactifications.

I. Preliminaries

This report subsumes a basic knowledge of general topology; nevertheless, we reserve the first chapter, in part, for recalling some of the results in general topology which will be useful in the sequel. We also introduce notation which will be followed throughout this paper. In addition, we provide proofs of several useful theorems of a general nature, but which are not usually included in a book on general topology. Proofs which can be found in [1] shall be omitted.

1.1 Notation. In a topological space $(X,\mathfrak{T})$ we use the following notation:

1) If $\mathfrak{T}$ is to remain fixed in a particular discussion, we denote the space by X, where there is understood to be a topology defined on X.

2) If $A \subset X$, then $\mathrm{Int}_X A$ denotes the <u>interior</u> of the set A in X. When no confusion can arise we denote $A^o = \mathrm{Int}_X A$.

3) If $A \subset X$, then $\mathrm{cl}_X(A)$ denotes the <u>closure</u> of the set A in X.

4) If $A \subset X$, then $\mathrm{Fr}_X A$ denotes the <u>frontier</u> (boundary) of the set A in X.

5) $\mathcal{P}(X)$ denotes the power set of X.

We will be concerned, in the sequel, with determining when a topological space is compact. We find the following characterization most useful.

1.2 <u>Theorem</u>. A topological space X is compact if and only if it has the finite intersection property: For each family $\{F_\alpha | \alpha \in \mathcal{A}\}$ of closed sets in Y satisfying $\bigcap_{\alpha\in\mathcal{A}} F_\alpha = \emptyset$, there is a finite subfamily $F_{\alpha_1}, \cdots, F_{\alpha_n}$ with $\bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset$.

We recall that every compact space is, in particular, normal; and that every subspace of a normal space is completely regular.

1.3 Definition. A topological space X is a continuum if X is compact and connected.

1.4 Definition. A map $f: S \to T$ is said to be monotone if $f^{-1}(t)$ is connected for every $t \in T$.

It is obvious that a continuous map $f: X \to Y$, where X is a compact topological space and Y is Hausdorff, is monotone if and only if $f^{-1}(y)$ is a continuum for each $y \in Y$.

1.5 Theorem. Every completely regular space X has at least one compactification. In particular, X has a Stone-Čech compactification, which we denote βX.

We now define an ordering on the set of all compactifications of a given completely regular space X.

1.6 Definition. If Y and Z are compactifications of the space X, we say Y precedes Z, and write $Y \leq Z$ or $Z \geq Y$, if there is a continuous surjection $\varphi: Z \to Y$ such that $\varphi|_X = 1_X$. In this case, we call the map φ the natural surjection from Z onto Y.

The ordering "$\geq$" in 1.6 is, in fact, a partial ordering for the set of all compactifications of X. It is a well-known fact that the Stone-Cech compactification βX, mentioned in 1.5, is the maximal (with respect to the partial ordering defined in 1.6) compactification of the space X.

1.7 Theorem. Every locally compact space X has a minimal compactification; namely, it has a one-point compactification, which we denote by ψX.

1.8 Theorem. If X and Y are topological spaces and $A \subset X \subset Y$, then we have the following:

1) $cl_X(A) \subset cl_Y(A)$,

2) $cl_X(A) = cl_Y(A) \cap X$, and

3) $cl_Y(A) = cl_Y(cl_X(A))$.

Since we will be concerned with completely regular spaces and their compactifications, it is important to have several criteria for the extendability of functions.

1.9 Theorem. Let X be a dense subset in each of the two Hausdorff spaces Y and Z, and let the identity map

$$1_X : X \to X$$

be extendable to a continuous $f : Y \to Z$ and also to a continuous $g : Z \to Y$. Then f and g are homeomorphisms, and $f = g^{-1}$.

Proof. Since the two continuous functions $g \circ f: Y \to Y$ and $1_Y: Y \to Y$ agree on the dense set X of the Hausdorff space Y, it follows from theorem 1.2,2) of ([1], VII), that $g \circ f = 1_Y$. Similarly, we have $f \circ g = 1_Z$. The conclusion follows from theorem 12.3 of ([1], III).

We recall the following useful theorem, and apply it to prove a theorem of Taimanov.

1.10 Theorem. Let D be a dense subset of X, let Y be a regular space, and $f: D \to Y$ be continuous. Then f has a continuous extension $F: X \to Y$ if and only if the filterbase

$$f(D \cap \mathfrak{U}(x))$$

converges for each $x \in X$. If F exists, then F is unique.

1.11 Theorem. (Taimanov) Let X be dense in Z. Then a necessary and sufficient condition that a continuous function f from X into the compact Hausdorff space Y have a continuous extension F from Z into Y is that for each two disjoint closed sets A and B in Y, $cl_Z(f^{-1}[A])$ and $cl_Z(f^{-1}[B])$ be disjoint.

Proof. Necessity is obvious.

Sufficiency: Suppose $f: X \to Y$ is a continuous map. If $z \in Z$, let $\mathcal{U}(z)$ be the open neighborhood filterbase at z. Then $\mathcal{U}(z) \cap X$ is a filterbase in X, and

$$\mathcal{M}(z) = \{cl_Y(f[U \cap X]) \mid U \in \mathcal{U}(z)\}$$

is therefore a filterbase in Y. Since Y is compact, and since the intersection of any finite subfamily of $\mathcal{M}(z)$ is non-empty, then by 1.2, we have:

$$M_z = \bigcap \{cl_Y(f[U \cap X]) \mid U \in \mathcal{U}(z) \neq \emptyset.$$

We show that M_z is a singleton. Suppose s and t are distinct points in M_z. By the regularity of Y, we can find disjoint closed neighborhoods A and B of s and t, respectively. By the condition of the theorem, we have:

$$(1) \qquad cl_Z(f^{-1}[A]) \cap cl_Z(f^{-1}[B]) = \emptyset.$$

On the other hand, if $x \in M_z$ and K is any neighborhood (not necessarily open) of x, then for every $U \in \mathcal{U}(z)$, $f(U \cap X) \cap K \neq \emptyset$, and hence $U \cap f^{-1}[K] \neq \emptyset$. Thus every open neighborhood of z intersects $f^{-1}[K]$, so that

$$z \in cl_Z(f^{-1}[K])$$

for every neighborhood (not necessarily open) of any point $x \in M_z$. Hence it follows that

(2) $$z \in [\mathrm{cl}_Z(f^{-1}[A]) \cap \mathrm{cl}_Z(f^{-1}[B])].$$

Since (2) is contrary to (1), then M_z must be a singleton. Thus $\mathcal{M}(z)$, and therefore $f(\mathcal{U}(z) \cap X)$, converges. The conclusion now follows from 1.10.

1.12 Theorem. If Y and Z are distinct (i.e., non-homeomorphic) Hausdorff compactifications of the space X, then there are sets $A, B \subset X$ with $\mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(B) = \emptyset$ whereas $\mathrm{cl}_Z(A) \cap \mathrm{cl}_Z(B) \neq \emptyset$, or with $\mathrm{cl}_Z(A) \cap \mathrm{cl}_Z(B) = \emptyset$ whereas $\mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(B) \neq \emptyset$.

Proof. This follows directly from 1.9 and 1.11.

As we are applying the theory of proximity spaces to the study of compactifications (see Chapter V), we shall have need of some basic dimension theory.

1.13 Definition. A space X is zero-dimensional, ind $X = 0$, if for every point $p \in X$, and every open neighborhood $U(p)$ there is an open set V with $p \in V \subset U(p)$ and with $\mathrm{Int}_X V = \emptyset$.

1.14 Definition. The weight of a topological space is the least cardinal of a basis of the topology on X.

1.15 Definition. A mapping $f\colon S \to T$ is said to be zero-dimensional (light) if for every point $t \in T$, $f^{-1}(t)$ is a totally disconnected subset of S.

II. Proximity Spaces

The notion of proximity spaces was introduced by V. A. Efrimove and had been shown useful by mathematicians including: Efrimovic Yu M. Smirnov, and E. G. Skljarenko. In the present chapter we limit ourselves to the task of formulating a proximity theory similar to that of Efrimovic, and to deriving the various inherent properties of proximities.

2.1 <u>Definition</u>. A <u>proximity space</u> consists of a pair (P,δ), where P is a non-empty set, and δ is a relation on $\mathcal{P}(P)$, called a <u>proximity relation</u>, and such that the following properties are satisfied:

1) $A \ \delta \ B$ iff $B \ \delta \ A$.
2) $(A \cup B) \ \delta \ C$ iff $A \ \delta \ C$ or $B \ \delta \ C$.
3) For any points $x,y \in P$, $\{x\} \ \delta \ \{y\}$ iff $x = y$.
4) $A \not\delta \ \emptyset$ holds for all $A \subset P$, where $\not\delta$ denotes the negation of δ.
5) If $A \not\delta \ B$, then there exist sets C, $D \subset P$ such that $C \cup D = P$, $A \not\delta \ C$, and $B \not\delta \ D$.

<u>Remarks</u>. We shall say that A is close to (far from) B if $A \ \delta \ B$($A \not\delta \ B$,resp.). Moreover, when no confusion can arise, we shall say that P is a <u>δ-space</u>, where δ is understood to be the proximity relation on the set $\mathcal{P}(P)$. Finally, for convenience, we shall consider the relation δ as a function from the set $\mathcal{P}(P) \times \mathcal{P}(P)$ into the subset $\{0,1\}$ of the reals. The function is defined as follows:

$$\delta(A,B) = \begin{cases} 1 & \text{if } A \not\delta \ B \text{, and} \\ 0 & \text{if } A \ \delta \ B. \end{cases}$$

In this notation, property 2) is equivalent to:

2') $$\delta(A \cup B,C) = \min\{\delta(A,C),\ \delta(B,C)\}.$$

In fact, we can do even better than 2').

2.2 Proposition. $\delta(\bigcup_{i=1}^{n} A_i, B) = \min_{i=1,\cdots,n} \{\delta(A_i,B)\}$.

Proof. We prove that 2') is equivalent to 2), so that 2.2 follows from 2') by induction of the natural number n. Suppose a proximity relation δ satisfies property 2). If $\delta(A \cup B,C) = 1$, then 2) shows that $\delta(A,C) = 1$ and $\delta(B,C) = 1$, so $\min\{(A,C),(B,C)\} = 1$. If $\delta(A \cup B,C) = 0$, then from 2) it follows that either $\delta(A,C) = 0$ or $\delta(B,C) = 0$, so that $\min\{\delta(A,C),\delta(B,C)\} = 0$. Hence 2') follows from 2). Suppose 2') holds. From 2') we have that $\delta(A \cup B,C) = 0$ iff $\min\{\delta(A,C),\delta(B,C)\} = 0$. But $\min\{\delta(A,C),\ \delta(B,C)\} = 0$ iff $\delta(A,C) = 0$ or $\delta(B,C) = 0$. Thus 2) is satisfied. The proposition is proved.

With the definitions and notation just developed, we will prove three useful properties of any δ-space P.

2.3 Proposition. If A, B $\subset$ P and A $\subset$ B, then for any C $\subset$ P, we have $\delta(A,C) \geq \delta(B,C)$.

Proof. The proof conveniently divides into the cases where C is close to B and where C is far from B. In case $\delta(B,C) = 0$, then $\delta(A,C) = 0$ or 1, so $\delta(A,C) \geq \delta(B,C)$. If $\delta(B,C) = 1$, then $\delta(B,C) = \delta(A \cup B,C) = \min\{\delta(A,C),\delta(B,C)\} = 1$; so that $\delta(A,C) = 1$, and $\delta(A,C) \geq \delta(B,C)$. The proposition is proved.

2.4 Proposition. Any sets which intersect are close.

Proof. Let A, B $\subset$ P such that $A \cap B \neq \emptyset$. Then there is a point $x \in A \cap B$, and by 3) $\delta(x,x) = 0$. Since $\{x\} \subset A$, then 2.3 gives $\delta(x,A) = 0$. Since $\{x\} \subset B$, then 2.3 gives $\delta(B,A) = 0$.

2.5 Proposition. No set is close to the empty set.

Proof. This is precisely the contrapositive of 4) in definition 2.1

Now that the notion of a proximity space has been defined, it is reasonable to try finding the relationship between this new type of space and those already known. For instance, one should look for structures in the δ-spaces. Perhaps there is a natural topological structure lurking in the background. If so, it would be helpful to discover what kind of topologies can be induced by proximity relations.

A very natural way of inducing a topology in a δ-space P is to call a set $A \subset P$ closed iff it contains all points of P which are close to it under the given proximity relation δ. Having determined the closed sets in the space P, we also know which sets are open; of course we don't know, as yet, whether or not this collection of open sets forms a topology for the set P. In the sequel, we shall denote the above mentioned collection of open sets by $T(\delta)$; that is, $T(\delta) = \{U \subset P \mid U = P - A, \text{ and } A \text{ is closed}\}$.

2.6 Proposition. $T(\delta)$ forms a topology for the δ-space P.

Proof. Since P contains all the points, then P is closed; so that, $P - P = \emptyset \in T(\delta)$. No point is close to $\emptyset$, so $\emptyset$ is closed; and $P - \emptyset = P \in T(\delta)$. To conclude the proof of 2.6, we show that all finite unions and arbitrary intersections of closed sets in P are also closed in P. Suppose $\{A_i \mid i=1,\cdots,n\}$ is a collection of closed sets in P. If $x \in P$ such that $\delta(x, \bigcup_{i=1}^{n} A_i) = 0$, then it follows from 2') of definition 2.1, that $\min_{i=1,\cdots,n} \{\delta(x,A_i)\} = 0$; hence $\delta(x,A_j)=0$, for some $j \in \{1, \cdots ,n\}$. Since A_j is closed, then $x \in A_j$; and hence

$x \in \bigcup_{i=1}^{n} A_i$. Thus, $\bigcup_{i=1}^{n} A_i$ is a closed subset of P. If $\mathcal{A}$ is an arbitrary indexing set, let $\{A_\alpha | \alpha \in \mathcal{A}\}$ denote a collection of closed subsets of P. If $x \in P$ with $\delta(x, \bigcap_{\alpha \in \mathcal{A}} A_\alpha) = 0$, then $\delta(x, A_\beta) = 0$ for each $\beta \in \mathcal{A}$, since $\bigcap_{\alpha \in \mathcal{A}} A_\alpha \subset A_\beta$ for each $\beta \in \mathcal{A}$. Since A_β is closed for each $\beta \in \mathcal{A}$, then $x \in A_\beta$ for each $\beta \in \mathcal{A}$; hence $x \in \bigcap_{\alpha \in \mathcal{A}} A_\alpha$. Thus $\bigcap_{\alpha \in \mathcal{A}} A_\alpha$ is also a closed set in P.

<u>Note</u>. To distinguish $T(\delta)$ from other topologies which may be defined on the set P, we shall call $T(\delta)$ the topology on P induced by the proximity relation δ. Since $T(\delta)$ is a topology, it has a closure operation associated with it; and we denote this operation by $cl_\delta(\ )$.

In an attempt to find relationships between proximity spaces and topological spaces, we now prove two lemmas relating the proximity relation δ to the closure operation $cl_\delta(\ )$.

2.7 <u>Lemma</u>. If P is a δ-space and $A \subset P$, then $cl_\delta(A) = \{x \in P | \delta(x,A) = 0\}$.

<u>Proof</u>. Denote $A^* = \{x \in P | \delta(x,A) = 0\}$. We wish to prove that $cl_\delta(A) = A^*$. If $x \in A^*$, then $\delta(x,A) = 0$. Since $A \subset cl_\delta(A)$, then we have $\delta(x, cl_\delta(A)) = 0$. But $cl_\delta(A)$ is closed, so that $x \in cl_\delta(A)$. Thus, we have $A^* \subset cl_\delta(A)$. In any topological space, $cl_x(A) = \bigcap \{A_\alpha | A_\alpha \supset A,\ A_\alpha \text{ is closed}\}$. We show $cl_\delta(A) \subset A^*$ by proving A^* to be one of the sets in the intersection which forms $cl_\delta(A)$. Clearly, $A^* \supset A$, so we show A^* is closed. Take any $x \notin A^*$, then $\delta(x,A) = 1$. So there are sets $C, D \subset P$ with $C \cup D = P$, and $\delta(x,D) = 1 = \delta(A,C)$. Every point of C is far from A, so that $A^* \subset P - C \subset D$. Since $\delta(x,D) = 1$, and $A^* \subset D$, then $\delta(x,A^*) = 1$. We have shown that every point close to A^* is contained in A^*, so that A^* is indeed closed in P. Thus, $cl_\delta(A) = A^*$; and the lemma is proven.

2.8 Lemma. If A, B $\subset$ P, then $\delta(A,B) = \delta(cl_\delta(A), cl_\delta(B))$.

Proof. If $\delta(A,B) = 0$, then $A \subset cl_\delta(A)$ and $B \subset cl_\delta(B)$; so that by 2.3, we also have $\delta(cl_\delta(A), cl_\delta(B)) = 0$. In the case where $\delta(A,B) = 1$, then we have sets $C,D \subset P$ with $C \cup D = P$ and $\delta(A,C) = 1 = \delta(B,D)$. Since $\delta(A,C) = 1$, then no point close to A is in C; so that $cl_\delta(A) \subset P - C \subset D$. $\delta(B,D) = 1$, so by 2.3 we have $\delta(B, cl_\delta(A)) = 1$. Repeating the above process, we get $\delta(cl_\delta(B), cl_\delta(A)) = 1$. Thus we always have $\delta(A,B) = \delta(cl_\delta(A), cl_\delta(B))$, and 2.8 is proven.

It now becomes necessary to introduce a concept of "neighborhoods" in δ-spaces. This concept will turn out to be an important link in connecting proximity spaces with topological spaces.

2.9 Definition. If A, B $\subset$ P, then B is called a δ-neighborhood of A in the δ-space P if A is far from P - B, and we write $B \supset\!\!\!\supset A$.

It will be useful to restate property 5) of definition 2.1 in terms of δ-neighborhoods:

5') Any two sets A, B $\subset$ P which are far from one another, have disjoint δ-neighborhoods.

2.10 Proposition. In the presence of the first four properties of definition 2.1, 5') is equivalent to 5) of 2.1.

Proof. Suppose P is a δ-space as defined in 2.1, and A, B $\subset$ P with $\delta(A,B) = 1$. By 5) of 2.1, we have subsets C and D of P such that $C \cup D = P$ and $\delta(A,C) = 1 = \delta(B,D)$. Clearly C - D and D - C are disjoint δ-neighborhoods of the sets A and B, respectively, so that 5') is satisfied. If, on the other hand, properties 1) - 4) of 2.1 are satisfied and 5') is satisfied, then for any sets A, B $\subset$ P which are far apart, there exist disjoint δ-neighborhoods $A^*, B^* \subset P$

of the sets A and B, respectively. Let $C = B^*$ and $D = P - B^*$. By definition of a δ-neighborhood,

(1) $$\delta(D,B) = 1.$$

Since $P - A^* \supset B^*$ and $\delta(P - A^*, A) = 1$, then $\delta(B^*, A) = 1$; so that

(2) $$\delta(C,A) = 1$$

Since $C \cup D = B^* \cup P - B^*$, then

(3) $$C \cup D = P.$$

(1), (2), and (3) clearly demonstrate that property 5) is satisfied. Thus the proof of the proposition is complete.

2.11 Proposition. In a given δ-space P, the following properties of δ-neighborhoods always hold:

1) If $A, B \subset P$ and $B \Supset A$, then $P - A \Supset P - B$.

2) If $A, B \subset P$ and $B \Supset A$, then $B \supset A$.

3) If $A, B, C \subset P$ and i) $C \Supset B \supset A$ or ii) $C \supset B \Supset A$, then $C \Supset A$.

4) If $B_i \Supset A_i$ where $i - 1, \cdots, k$; then both $\bigcup_{i=1}^{k} B_1 \Supset \bigcup_{i=1}^{k} A_i$ and $\bigcap_1^k B_1 \Supset \bigcap_1^k A_i$.

5) If $A, C, \subset P$ and $C \Supset A$, then there is a set $B \subset P$ such that $C \Supset B \Supset A$.

Proof of 1). If $B \Supset A$, then $\delta(P - B, A) = 1$. But $A = P - (P - A)$, so $\delta(P - B, P - (P - A)) = 1$; which proves that $P - A \Supset P - B$.

Proof of 2). Suppose $B \Supset A$. If $x \in A$, then $x \notin P - B$, because $\delta(P - B,A) = 1$. Since $x \in P$, then it must be that $x \in B$; which proves that $B \supset A$.

<u>Proof of 3)</u>. If i) holds, then $\delta(P - C, B) = 1$. Since $B \supset A$, then by 2.3 $\delta(P - C,A) = 1$. Thus $C \supset\!\!\!\supset A$. If ii) holds, then $\delta(P - B,A) = 1$. But $C \supset B$, so $P - C \subset P - B$. Hence by 2.3, $\delta(P - C,A) = 1$; and so $C \supset\!\!\!\supset A$.

<u>Proof of 4)</u>. Suppose $B_i \supset\!\!\!\supset A_i$ for i - 1, $\cdots$, k.

<u>Claim 1</u>: $(B_1 \cup B_2) \supset\!\!\!\supset (A_1 \cup A_2)$.

<u>Proof of claim 1</u>. Since $B_1 \supset\!\!\!\supset A_1$ and $B_2 \supset\!\!\!\supset A_2$, then $\delta(P - B_1,A_1) = 1 = \delta(P - B_2,A_2)$. $P - (B_1 \cup B_2) = (P - B_1) \cap (P - B_2)$, and $P - B_1 \supset (P - B_1) \cap (P - B_2)$, and $P - B_2 \supset (P - B_1) \cap (P - B_2)$; so by 2.3 we have that $\delta([(P - B_1) \cap (P - B_2)],A_1) = 1$ and $\delta([(P - B_1) \cap (P - B_2)], A_2) = 1$. Hence by property 2) of definition 2.1 we have that $\delta([(P - B_1) \cap (P - B_2)],[A_1\ A_2]) = 1$; that is, $\delta([P - (B_1 \cup B_2)],[A_1 \cup A_2]) = 1$. This proves that $(B_1 \cup B_2) \supset\!\!\!\supset (A_1 \cup A_2)$, which was claim 1.

<u>Claim 2</u>: $(B_1 \cap B_2) \supset\!\!\!\supset (A_1 \cap A_2)$.

<u>Proof of claim 2</u>. Since $B_1 \supset\!\!\!\supset A_1$ and $B_2 \supset\!\!\!\supset A_2$, then $\delta(P - B_1,A_1) = 1 = \delta(P - B_2,A_2)$. $A_1 \cap A_2 \subset A_1$ and $A_1 \cap A_2 \subset A_2$, so be 2.3 $\delta(P - B_1,A_1 \cap A_2) = 1 = d(P - B_2, A_1 \cap A_2)$. By part 2) of definition 2.1 $\delta([(P - B_1) \cup (P - B_2)], [A_1 \cap A_2]) = 1$. That is, $\delta([P - (B_1 \cap B_2)], [A_1 \cap A_2]) = 1$, which proves claim 2.

Part 4) of 2.11 follows from claims 1 and 2 by induction.

<u>Proof of 5)</u>. Suppose $C \supset\!\!\!\supset A$, then $\delta(P - C,A) = 1$. So there exist disjoint δ-neighborhoods B and D of the sets A and P - C respectively. Thus $\delta(P - D, P - C) = 1$, and since $B \subset P - D$, then $\delta(B, \beta - C) = 1$; hence $C \supset\!\!\!\supset B$. Since by construction $B \supset\!\!\!\supset A$, then we have $C \supset\!\!\!\supset B \supset\!\!\!\supset A$ as required in 5).

Thus we have completed the proof of the proposition.

<u>Example</u>. Let $R^* = [-\infty,\infty]$ with its usual topology. We define a proximity relation, δ, on R^* by: If $A, B \subset R^*$, then $A \delta B$ iff $cl_{R^*}(A) \cap cl_{R^*}(B) \neq \emptyset$. It

can be shown that $T(\delta)$ is exactly the usual topology on R^*. It is easy to see that the sets $(0,4), (0,4], [0,4)$, and $[0,4]$ are all δ-neighborhoods of the set $(1,2)$.

As the above example shows, δ-neighborhoods need not be open or closed, though they may be. We can however prove some relationships between δ-neighborhoods and open sets. We shall prove some of these relationships, and this will in turn give us some information about the topology, $T(\delta)$, induced by the proximity relation δ on a set P.

2.12 Lemma. For any δ-neighborhood $B \supseteq A$, there is an open set U such that $B \supset U \supseteq A$.

Proof. Since $B \supseteq A$, then $\delta(P - B, A) = 1$. From Lemma 2.8, we have $\delta(cl_P(P - B), A) = 1$. Since $cl_P(P - B)$ is closed, then $U = P - cl_P(P - B)$ is open. Then $P - U$ is far from A, so that $U \supseteq A$. Also, we have that

$$B = P - (P - B) \supset P - cl_P(P - B) = U.$$

Thus we have the open set U with $B \supset U \supseteq A$, and the lemma is proved.

2.13 Lemma. If A is a subset of the δ-space P, then the intersection of all the δ-neighborhoods of A is the closure of A, $cl_\delta(A)$.

Proof. We denote $A^* = \bigcap \{B \subset P \mid B \supseteq A\}$. $\delta(P - B, A) = 1$ for every $B \supseteq A$, so by lemma 2.8 we also have $\delta(P - B, cl_\delta(A)) = 1$. Thus $B \supseteq cl_\delta(A)$ whenever $B \supseteq A$, which proves $cl_\delta(A) \subset A^*$. If $x \notin cl_\delta(A)$, then $\delta(x, cl_\delta(A)) = 1$; so there exist disjoint δ-neighborhoods B' of A and C of $\{x\}$. Hence $x \notin B' \supset A^*$, and $x \notin A^*$; so that we also have $A^* \subset cl_\delta(A)$. Thus 2.13 is proved.

Proposition 2.6 shows that every δ-space P has a topology $T(\delta)$ associated with it. There arise new questions. For example, which, if any, of the separation axioms are always satisfied by $T(\delta)$? Can we place restrictions on δ

which force the topology $T(\delta)$ to be very specific (e.g. compact, metrizable, etc.)? Conversely, given a topological space X, is it always possible to define a proximity relation on X which is in some sense compatible with the topological structure of X? We can give a partial answer to the first question at this time, but the second question must wait for more machinery to be developed. Also, we can give a rather complete answer to the third question after we determine in what sense we want a proximity relation to be compatible with the topological space on which the relation is defined.

2.14 Proposition. If (P,δ) is a proximity space, then $T(\delta)$ is a Hausdorff topology.

Proof. We already know by proposition 2.6 that $T(\delta)$ is a topology. If $x,y \in P$ such that $x \neq y$, then by part 3) of definition 2.1 we have $\delta(x,y) = 1$. By 5') of definition 2.1 there exist disjoint δ-neighborhoods of $\{x\}$ and $\{y\}$, say $A \Supset \{x\}$ and $B \Supset \{y\}$. Applying lemma 2.12 we get open sets U and V in P such that $A \supset U \Supset \{x\}$ and $B \supset V \Supset \{y\}$. Since A and B are disjoint then so are U and V, so that $T(\delta)$ is Hausdorff.

2.15 Definition. A proximity space, (P,δ) is consistent with a topological space, $(R,\mathcal{T})$, if: i) $P = R$, and ii) $T(\delta) = \mathcal{T}$.

2.16 Theorem. For any compact space there is exactly one δ-space consistent with it.

Proof. It is sufficient to show:

(*) In any δ-space P consistent with the compact space R, sets $A, B \subset P$ are far apart if and only if $cl_R(A) \cap cl_R(B) = \emptyset$.

This is sufficient because the topology on R is completely determined by the closure operation $cl_R(\ )$, and (*) shows that the closure operation in R also completely determines the proximity relation δ.

Proof of (*). By lemma 2.8, $\delta(A,B) = \delta(cl_R(A), cl_R(B))$, so we need only prove (*) for closed sets $A, B \subset P$. If A and B are closed subsets of P with $\delta(A,B) = 1$, then by proposition 2.4 we have $A \cap B = \emptyset$. Hence we only have left to prove the converse of (*).

Suppose A and B are closed subsets of P such that $A \cap B = \emptyset$. Since B is closed and disjoint from A, then every point $x \in A$ is far from B. By property 5') of definition 2.1 there exist for every $x \in A$ disjoint δ-neighborhoods C_x of $\{x\}$ and D_x of B. By lemma 2.12 there is for each $x \in A$ an open set U_x such that $C_x \supset U_x \supseteq \{x\}$. $U_x \subset C_x \subset (P - D_x)$ and $\delta(P - D_x, B) = 1$, so by proposition 2.3 $\delta(U_x,B) = 1$. $\{U_x \mid x \in A\}$ is an open cover of A, and A is compact (since it is a closed subset of the compact space R). Hence there is a finite subcover, say $\{U_x \mid i = 1, \cdots, n\}$ of A; so that $\bigcup_{i=1}^{k} U_{x_i} \supset A$. Applying 2') of definition 2.1 we find that $\delta(\bigcup_{i=1}^{k} U_{x_i} B) = \min_{i=1,\ldots,k} \{\delta(U_{x_i},B)\} = \min_{i=1,\ldots,k} \{1\} = 1$. Since $\bigcup_{i=1}^{k} U_{x_i} \supset A$, then 2.3 implies that $\delta(A,B) = 1$. Thus (*) is proven.

Theorem 2.16 can be improved in some respects. It can be shown that even for a completely regular space there is a proximity space consistent with it; however, there may be many proximity relations which induce the same topology. First we prove a lemma which allows us to treat every subset of a proximity space as a proximity space itself. The induced proximity relation $\hat{\delta}$ on the subset Q of the δ-space P is defined as follows: If $A, B \subset Q$, then $\hat{\delta}(A,B) = 0$ if and only if $\delta(A,B) = 0$, where A and B are considered as subsets of P.

2.17 Lemma. If P is a δ-space and $Q \subset P$ with the induced proximity relation $\hat{\delta}$, then Q is a proximity space.

Proof. All five of the properties of a proximity space listed in definition 2.1 are clearly satisfied by the set Q with relation $\hat{\delta}$, as can be shown by restricting the corresponding properties of the space (P,δ) to the subset Q. Thus 2.17 is proven.

2.18 Theorem. For any completely regular space there is at least one δ-space consistent with it.

Proof. Let R be any completely regular space. By theorem 1.5, R has at least one compactification, say αR. By theorem 2.16 there is exactly one δ-space P_α, consistent with αR. Since $R \subset \alpha R$, then by lemma 2.17 R is a proximity space with the induced proximity $\hat{\delta}$.

Claim 1: The topology on R induced by the proximity relation $\hat{\delta}$ is consistent with that of R.

Proof of claim 1. $A \subset R$ is closed in the topology induced by the relation $\hat{\delta}$ iff A contains all points of R which are close to it under the proximity relation δ in the space αR. This means that if we denote the set of all points from αR which are close to A by $\bar{A}$, then $A = \bar{A} \cap R$. But $\bar{A}$ is a closed set in αR so $A = \bar{A} \cap R$ is precisely the statement that A is closed in the space R. Thus the topology induced by δ does coincide with that of the space R.

With claim 1, and the remarks preceeding that claim, the proof of theorem 2.18 is complete.

Observe that the proof of 2.18 gives more information than is stated in the theorem. For example, it follows that for each completely regular space R, there are at least as many δ-spaces consistent with it as there are non-homeomorphic compactifications of R. Of course, we have no way of knowing, at the present, whether these various proximities are distinct. Moreover, there arises a new question; namely, what is the relationship between the proximity spaces consistent with a given completely regular space and the compactifications of that space? These questions, and some more which are induced by answering these, will be taken up in chapters three and four.

It is obvious that we have been working in a category of objects which we called proximity spaces. In our study we have raised questions concerning the relationships of this category to other categories, such as the category of completely regular spaces. In this treatment, we do not pursue a categorical approach to the problems confronting us; but rather leave the category theory lurking in the background. Our only purpose in bringing the idea up was to stress the naturality in considering our next topic; that is, in considering the morphisms between the objects in the category of proximity spaces. Since δ-spaces are in fact topological spaces, then we call the morphisms from one δ-space into another mappings, as in usually done in topological spaces. Just as in groups we consider mappings which preserve the group operation, and in rings we consider maps which preserve both ring operations, and in topological spaces we consider maps which preserve points of adherence; then in proximity spaces we consider mapping which preserve proximity.

2.19 Definition. A map $f: P \to Q$, where P and Q are δ-spaces, is called a δ-map if and only if for any close subsets A and B of P then the images $f(A)$ and $f(B)$ are close in Q.

2.20 Proposition. Let P and Q be proximity spaces with proximity relations δ and $\hat{\delta}$, respectively. A function $f: P \to Q$ is a δ-map if and only if for any subsets A and B of Q such that $\hat{\delta}(A,B) = 1$, then $\delta(f^{-1}(A), f^{-1}(B))=1$.

Proof. Suppose $f: P \to Q$ is a δ-map. Let $A, B \subset Q$ such that $\hat{\delta}(A,B) = 1$. If

$\delta(f^{-1}(A), f^{-1}(B)) = 0$, then since f is a δ-map we also have $\hat{\delta}(f(f^{-1}(A)), f(f^{-1}(B)) = 0$. But $f(f^{-1}(A)) \subset A$ and $f(f^{-1}(B)) \subset B$, so by 2.3 $\delta(A,B) = 0$. This is a contradiction, so $\delta(f^{-1}(A), f^{-1}(B)) = 1$. Conversely, suppose that for every pair fo sets $A, B \subset Q$ which are far apart, it follows that $f^{-1}(A)$ is far from $f^{-1}(B)$. Let C and D be any subsets of P such that $\delta(C,D) = 0$. On the assumption that $\hat{\delta}(f[C], f[D]) = 1$, we find that $\delta(f^{-1}(f[C]), f^{-1}(f[D]) = 1$. But $C \subset f^{-1}(f[C])$ and $D \subset f^{-1}(f[D])$, so by 2.3 $\delta(C,D) = 1$ which is an obvious contradiction. This concludes the proof of 2.20.

2.21 Lemma. Any δ-map $f: P \to Q$ is continuous.

Proof. Let A be a closed subset of Q. If $x \in P$ such that $\delta(x, f^{-1}(A)) = 0$, then $\hat{\delta}(f(x), f(f^{-1}(A))) = 0$ because f is a δ-map. But $f(f^{-1}(A)) \subset A$, so by 2.3 $\hat{\delta}(f(x), A) = 0$. Since A is closed, then $f(x) \in A$. Hence $x \in f^{-1}(A)$, which concludes the proof that $f^{-1}(A)$ is closed. Thus, f is a continuous function.

2.22 Theorem. If $f: R \to Q$ is a continuous map, R is a compact space, and Q is a δ-space; then f is a δ-map.

Proof. Since R is compact, then by theorem 2.16 there is exactly one δ-space consistent with R, and in that δ-space two subsets are close iff their closures intersect. So if $A, B \subset R$ and $\delta(A,B) = 0$, then there is a point $x \in [cl_R(A) \cap cl_R(B)]$. Since f is continuous, then $f[cl_R(A)] \subset cl_{\hat{\delta}}(F[A])$ and $f[cl_R(B)] \subset cl_{\hat{\delta}}(f[B])$. Thus $f(x) \in (f[cl_R(A)] \cap f[cl_R(B)])$, and hence $f(x) \in [cl_{\hat{\delta}}(f[A]) \cap cl_{\hat{\delta}}(f[B])]$. By proposition 2.4, we have $\hat{\delta}(cl_{\hat{\delta}}(f[A]), cl_{\hat{\delta}}(f[B])) = 0$, where $\hat{\delta}$ is the proximity relation on Q. By 2.8 we also have that $\hat{\delta}(f[A], f[B]) = 0$. Hence f is a δ-map.

III. Extending Proximity Spaces.

After defining what we meant by a proximity space, we attempted to characterize δ-spaces in terms of some known topological concepts. We arrived at the fact that every proximity space (P,δ) induces a topology on the set P. The induced topology, which we denoted by $T(\delta)$, turned out never to be more general than a Hausdorff topology. We also found that if we are given any completely regular space $(R,\mathfrak{T})$, we can define at least one proximity relation δ, on R, such that the proximity space (R,δ) is consistent with the topological space $(R,\mathfrak{T})$.

In this chapter we shall extend and strengthen some of the results of chapter two. We will prove that the topology induced by any proximity space is more specific than Hausdorff; that it is, in fact, completely regular at the least. Combined with the information in the preceeding paragraph, we will have that the class of δ-spaces coincides with the class of completely regular spaces; and problems concerning completely regular spaces are translated to corresponding problems in δ-spaces. The advantages of this translation are obvious, since completely regular spaces are defined in terms of real-valued functions, which can become unruly; whereas, δ-spaces are defined by a relation δ with very simple properties. This, we hope, will give us insight, and possibly solutions, to problems of completely regular spaces.

To accomplish the proof that every proximity space is completely regular, we construct for every proximity spaces (P,δ), another proximity space $(uP,\hat{\delta})$, which contains the given space (P,δ) as a dense subset. The space $(uP,\hat{\delta})$ will turn out to be compact; which, of course, implies that (P,δ) is completely regular.

3.1 <u>Definition</u>. A δ-space $\tilde{P}$ is called a <u>δ-extension</u> of the δ-space P if $\tilde{P}$

contains P as an everywhere dense subspace.

3.2 Definition. A δ-space P is said to be absolutely closed if it has no δ-extensions except for itself; that is, if P is closed in every δ-space containing it.

The last remark in 3.2 reminds us that a compact (Hausdorff) space R is closed in every (Hausdorff) space containing it. This observation proves the following.

3.3 Proposition. Every compact space is absolutely closed.

We now ponder the converse of 3.3. If the converse were true, then our problem of embedding every δ-space in a compact space could be solved by embedding every δ-space in an absolutely closed δ-space. The converse is, in fact, true (cf. 3.25); and we begin the task of embedding an arbitrary δ-space (P,δ) into an absolutely closed δ-space.

Construction of uP for a fixed proximity space (P,δ).

3.4 Definition. In a δ-space P, a system of sets is said to be centered if the intersection of any finite number of them is not empty.

3.5 Definition. We call a system ξ of sets in a δ-space P a δ-system if each set $A \in \xi$ is a δ-neighborhood of some set $B \in \xi$.

At this point in the construction, we can consider the set of all centered δ-systems in a fixed δ-space P, but the resulting space turns out to be only a generalized δ-space. Hence we refine the collection of all centered δ-systems of the space P.

For a fixed centered δ-system ξ, in the fixed space (P,δ), we supplement ξ with the intersections of all finite subsystems of ξ, and call the new system ξ'.

3.6 Proposition. If ξ is a centered δ-system, then ξ' is a centered δ-system.

Proof. We break the proof into two claims. We first claim that ξ' is centered, and next that it is a δ-system.

Claim 1. ξ' is centered.

Proof of claim 1. Let $\{E_i | i2| , \cdots, n\}$ be a subsystem of ξ'. Each E_i is a finite intersection of member of ξ. So we may write for each i, $E_i = \bigcap_{j=1}^{k_i} E_{ij}$. Then $\bigcap_{i=1}^{n} E_i = \bigcap_{i=1}^{n} (\bigcap_{j=1}^{k_i} E_{ij})$ is still a finite intersection of members of ξ, so it is not empty since ξ is centered.

Claim 2. ξ' is a δ-system.

Proof of claim 2. Let $E \in \xi'$, then we must find a set $D \in \xi'$ and that $E \in D$. If $E \in \xi$, then the result is clear since ξ is a δ-system. If $E \bar{\in} \xi$ then $E = \bigcap_{i=1}^{n} E_i$ where each $E_i \in \xi$. ξ is a δ-system, so for each i, $3D_i \in \xi$ and $E_1 \supset D_i$. By 2.11 4) $E = \bigcap_{i=1}^{n} E_1 \supset \bigcap_{i=1}^{n} D_i$. This proves ξ' is a δ-system.

Claims 1 and 2 prove proposition 3.6.

Observe that the system ξ' has the property that it contains all intersections of finite subsystems of ξ'. This is true since every intersection of a finite number of members of ξ' is still a finite intersection of members of ξ.

We now supplement ξ' with all sets $A \subset P$ which contain a set $B \in \xi'$. We call this new system ξ'' and shall prove it is still a centered δ-system.

3.7 Proposition. ξ'' is a centered δ-system.

Proof. Again we break the proof into two parts; first proving ξ'' is centered and second proving ξ'' is a δ-system.

Claim 1: ξ'' is centered.

Proof of claim 1. Let $\{A_i | i = 1, \cdots, n\}$ be a subsystem of ξ''. If $A_i \in \xi'$, then $A_i \supset\!\!\!\supset B_i$ for some $B_i \in \xi'$. If $A_i \bar{\in} \xi'$, then, by construction, A_i contains some $B_i \in \xi'$. Hence in either case, for every A_i there is a $B_i \in \xi'$ such that $A_i \supset B_i$. Since ξ' is centered, then $\bigcap_{i=1}^{n} B_i \neq \emptyset$. $\bigcap_{i=1}^{n} A_i \supset \bigcap_{i-1}^{n} B_i$ so $\bigcap_{i-1}^{n} A_i \neq \emptyset$; which proves claim 1.

Claim 2: ξ'' is a δ-system.

Proof of claim 2. Suppose $A \in \xi''$. If $A \in \xi'$, then since ξ' is a δ-system, there is a $B \in \xi'$ and hence $B \in \xi''$ such that $A \supset\!\!\!\supset B$. If $A \bar{\in} \xi'$, then by construction, there is a set $C \in \xi'$ such that $A \supset C$. Again ξ' is a δ-system implies there is a $B \in \xi'$ and hence $B \in \xi''$ such that $C \supset B$. We now have a $B \in \xi''$ with $A \supset C \supset\!\!\!\supset B$ and by 2.11 3) this means $A \supset\!\!\!\supset B$. Hence for every $A \in \xi''$ there is a $B \in \xi''$ with $A \supset\!\!\!\supset B$. This proves claim 2.

The next step in the construction will be forming the elements of the set which will be a δ-extension of the original δ-space P. The elements of the δ-extension will be the maximal centered δ-systems from the δ-space P.

3.8 Definition. An end of the δ-space P is any centered δ-system, ξ, which is not a subsystem of any other centered δ-system. (Observe that an end is always a maximal filter.)

3.9 Proposition. If ξ is an end of the δ-space P, then every intersection of finite subsystems of ξ is still a member of ξ.

Proof. If ξ is an end for which the intersection of some finite subsystem is not a member of ξ, then we form a new system ξ' by supplementing ξ with all intersections of finite subsystems. By proposition 3.6, ξ' is also a centered δ-system. Since ξ' contains ξ properly, we have a contradiction to the fact that ξ is an end.

3.10 Proposition. If ξ is an end of the δ-space P, then every set $A \subset P$ which contains a member of ξ is also a member of ξ.

Proof. If this were not the case for some end ξ, then we could form a new system ξ' by supplementing ξ with all sets $A \subset P$ which contain some element of ξ. By proposition 3.7, ξ' is a centered δ-system. Since ξ' contains ξ properly, then we have a contradiction to the fact that ξ is an end.

3.11 Proposition. If P is a δ-space, then every centered δ-system of P is contained in some end of P.

Proof. We shall use Zorn's lemma to prove this proposition. First we define an ordering ">>" on the set, S, of all centered δ-systems containing a given centered δ-system ξ. If $\xi', \xi'' \in S$ then $\xi' >> \xi''$ iff ξ'' is a subsystem of ξ'. The fact that ">>" is reflexive and transitive follows from the corresponding properties of set inclusion. Thus ">>" is a preordering in S. Each chain in S certainly has an upper bound, since the union of all the elements of a given chain of members of S is easily shown to be a centered δ-system. Hence Zorn's lemma does apply, and there is a maximal element in the set S. This maximal element of S is clearly an end which contains the centered δ-system ξ.

3.12 An example of a centered δ-system is the set ξ_A of all δ-neighborhoods of a given nonempty set $A \subset P$. ξ_A is centered because every element of ξ_A

contains A and thus every intersection of a subsystem of ξ_A contains the non-empty set A. ξ_A is a δ-system since if $E \in \xi_A$, then $E \supset A$. By 2.11 5), there is a set $B \subset P$ such that $E \supset B \supset A$. So $B \in \xi_A$ and $E \supset B$.

We shall now prove the very important observation that if the set A in the above example consists of one point, then the centered δ-system ξ_A is an end of the δ-space P. This fact will enable us to establish a correspondence between the set of ends of the δ-space P, and the points of P.

3.13 <u>Lemma.</u> If x is an element of the δ-space P, then the centered δ-system $\xi_x = \{A \subset P | A \supset \{x\}\}$ is an end of the δ-space P.

<u>Proof.</u> In 3.12 we showed that ξ_x is in fact a centered δ-system, so to prove 3.13 we show: if ξ is a δ-system which properly contains ξ_x, then ξ is not centered. $\xi \supsetneqq \xi_x$ means there is a set $A \in \xi$ and $A \bar{\in} \xi_x$. ξ is a δ-system, so there is a set $B \in \xi$ such that $A \supset B$. By 2.11 5) there is a set $C \subset P$ with $A \supset C \supset B$. Since $A \bar{\in} \xi_x$, then $A \not\supset \{x\}$. $A \supset B$ and $A \not\supset x$ means $x \bar{\in} B$, so $x \in P-B$. $B \supset C$ so $\delta(P - B, C) = 1$ and so $\delta(x, C) = 1$. $C = P - (P - C)$ so $\delta(x, P - (P - C)) = 1$ which means that $P - C \supset \{x\}$. Hence $(P - C) \in \xi_x$. But $\xi_x \subset \xi$, so $(P - C) \in \xi$. Since $C \in \xi$, we have $C, P - C \in \xi$ and $C \cap (P - C) = \emptyset$. This means ξ is not centered, which concludes the proof of 3.12.

<u>Notation.</u> If P is a δ-space, we denote the set of all ends of P by <u>uP</u>.

3.14 <u>Definition.</u> $0 < > : 2^P \to 2^{uP}$, is defined as follows: If $A \subset P$, then $0 \langle A \rangle = \{\xi \in uP | A \in \xi\} \subset uP$.

3.15 <u>Definition.</u> $\hat{\delta}$ is a relation on the set of all subsets of uP, and is defined as follows: If $C, D \subset uP$, then $C \,\hat{\not\delta}\, D$ if and only if there exist sets $A, B \subset P$ with $A \not\delta B$, and such that $C \subset 0\langle A \rangle$, $D \subset 0\langle B \rangle$. That is,

$\hat{\delta}(C,D) = 1$, if and only if there exist sets A, $B \subset P$ with $\delta(A,B) = 1$ and $C \subset 0\langle A\rangle$, $D \subset 0\langle B\rangle$.

Before proving that the realtion in 3.15 is a proximity relation, we need to investigate the behavior of the operator $0\langle\ \rangle$, which carries subsets of P into subsets of uP.

3.16 Proposition. If P is a δ-space and A, $B \subset P$, then the following relationships hold:

1) $0\langle A\rangle \cap 0\langle B\rangle = 0\langle A \cap B\rangle$
2) $0\langle A\rangle \cup 0\langle B\rangle \subset 0\langle A \cup B\rangle$
3) If $\delta(P - A, P - B) = 1$, then $0\langle A\rangle \cup 0\langle B\rangle = uP$.

Note: The relation in 1) may be extended to a finite number of sets, and 2) may be extended to any number of sets.

Proof of 1). Suppose $\xi \in 0\langle A\rangle \cap 0\langle B\rangle$. Since $\xi \in 0\langle A\rangle$, then $A \in \xi$. Since $\xi \in 0\langle B\rangle$, then $B \in \xi$. $\xi \in uP$ so by 3.9 $(A \cap B) \in \xi$. Thus by definition 3.14 we have that $\xi \in 0\langle A \cap B\rangle$. Hence we have proven $0\langle A\rangle \cap 0\langle B\rangle \subset 0\langle A \cap B\rangle$. Suppose $\xi \in 0\langle A \cap B\rangle$, then $A \cap B \in \xi$. Since $\xi \in uP$ and $A \supset (A \cap B)$ and $B \supset (A \cap B)$, we have by 3.10 that $A \in \xi$ and $B \in \xi$. This proves $\xi \in 0\langle A\rangle$ and $\xi \in 0\langle B\rangle$. Hence $0\langle A \cap B\rangle \subset 0\langle A\rangle \cap 0\langle B\rangle$. Thus 1) is proven.

Proof of 2). Suppose $\xi \in 0\langle A\rangle \cup 0\langle B\rangle$. Then either $\xi \in 0\langle A\rangle$ or $\xi \in 0\langle B\rangle$. If $\xi \in 0\langle A\rangle$, then $A \in \xi$. Since $A \cup B \supset A$, then $A \cup B \in \xi$. But $A \cup B \in \xi$ means $\xi \in 0\langle A \cup B\rangle$. Similarly, $\xi \in 0\langle B\rangle$ implies $\xi \in 0\langle A \cup B\rangle$. Hence in any case $\xi \in 0\langle A\rangle \cup 0\langle B\rangle$ implies $\xi \in 0\langle A \cup B\rangle$. This concludes proof of 2).

Proof of 3). Clearly $0\langle A\rangle \cup 0\langle B\rangle \subset uP$. Suppose $\xi \in uP$. We want to show $\xi \in 0\langle A\rangle \cup 0\langle B\rangle$. In case $A = P$, then $A \in \xi$ because it is a trivial fact that

P is contained in every end. $A \in \xi$ implies $\xi \in 0\langle A\rangle$, so in this case $\xi \in 0\langle A\rangle \cup 0\langle B\rangle$. If $A \neq P$, then $P - A \neq \emptyset$. Let $D = P - A$, then from 3.11 it follows that $\xi_D = \{E \subset P \mid E \supset D\}$ is a centered δ-system. If some $H \in \xi$ is contained in A; that is, if H does not meet the set D, then $A \in \xi$ and so $\xi \in 0\langle A\rangle$, which proves $\xi \in 0\langle A\rangle \cup 0\langle B\rangle$. If there does not exist such an $H \in \xi$, then every $H \in \xi$, meets the set D. In this case we make the claim that $\xi_D \cup \xi$ is a centered δ-system.

Claim 1: If every $H \in \xi$ meets D, then $\xi_D \cup \xi$ is centered.

Proof of claim 1. Let $\{D_i \mid i = 1,\cdots,n\}$ be a finite subsystem of $\xi_D \cup \xi$. Suppose $D_i \in \xi_D$ for $i = 1,\cdots,k \leq n$ and that $D_i \in \xi$ for $i = k+1, k+2,\cdots,n$. So we can write

(1) $$\bigcap_{i=1}^{n} D_i = \left(\bigcap_{i=1}^{k} D_i\right) \cap \left(\bigcap_{i=k+1}^{n} D_i\right).$$

Let $E = \bigcap_{i=k+1}^{n} D_i$, then $E \in \xi$, because ξ is an end. Since $D_i \supset D$ for $i = 1,\cdots,k$ then also $D_i \supset D$ for $i = 1,\cdots,k$. Hence we have $\bigcap_{i=1}^{k} D_i \supset D$. Thus from (1) we have

(2) $$\bigcap_{i=1}^{n} D_i \supset D \cap E.$$

By hypothesis we know that $D \cap E \neq \emptyset$; so from (2) we have

(3) $$\bigcap_{i=1}^{n} D_i \neq \emptyset$$

(3) verifies that $\xi_D \cup \xi$ is centered.

Claim 2: If every $H \in \xi$ meets D, then $\xi_D \cup \xi$ is a δ-system.

Proof of claim 2. Let $A \in \xi_D \cup \xi$. If $A \in \xi_D$, then there is a $B \in \xi_D$ such that $A \supset B$, because ξ_D is a δ-system. Since $B \in \xi_D$, then also $B \in \xi_D \cup \xi$. If $A \in \xi$, then there is a $B \in \xi$, hence $B \in \xi_D \cup \xi$, such that $A \supset B$. Thus in either case if $A \in \xi_D \cup \xi$ then there is a $B \in \xi_D \cup \xi$ with $A \supset B$, which proves that $\xi_D \cup \xi$ is a δ-system.

Claims 1 and 2 show that $\xi_D \cup \xi$ is a centered δ-system. Since ξ is an end, then $\xi_D \subset \xi$. Since $D = P - A$, then we have by hypothesis that $\delta(D, P - B) = 1$, so that $B \supset D$. This means that $B \in \xi_D$. Since $\xi_D \subset \xi$, then $B \in \xi$. This means $\xi \in 0\langle B\rangle$ and so $\xi \in 0\langle A\rangle \cup \langle B\rangle$. Since we have shown that an arbitrary element $\xi \in uP$ is also contained in $0\langle A\rangle \cup 0\langle B\rangle$, then we have $uP \subset 0\langle A\rangle \cup 0\langle B\rangle$. Hence we have proven part 3) of 3.16.

Having proven some properties of the operator $0 \langle\ \rangle$, we will now apply these properties in demonstrating that the relation defined in 3.15 is actually a proximity relation on uP.

3.17 Proposition. The relation $\hat{\delta}$ defined in 3.15 on the set uP of ends of the δ-space P, is a proximity relation.

Proof. The proof of this proposition consists of verifying that $\hat{\delta}$ satisfies the five properties of a proximity relation as set forth in definition 2.1.

Verification of 1). Suppose $C, D \subset uP$, and $\hat{\delta}(C,D) = 1$. Then there must be sets $A,B \subset P$ such that $\delta(A,B) = 1$ and $C \subset 0\langle A\rangle$, while $D \subset 0\langle B\rangle$. Since $\delta(A,B) = \delta(B,A)$, and B we get $\hat{\delta}(D,C) = 1$.

Verification of 2). We shall verify the contrapositive of 2): $\hat{\delta}(C \cup D,E) = 1$ if and only if $\hat{\delta}(C,E) = 1$ and $\hat{\delta}(D,E) = 1$. If $C,D,E \subset uP$ and $\hat{\delta}(C \cup D,E) = 1$, then there are sets $A,B \subset P$ with $\delta(A,B) = 1$ and $C \cup D \subset 0\langle A\rangle$, $E \subset 0\langle B\rangle$.

Hence it is true that $C \subset 0 \langle A \rangle$ and $D \subset 0 \langle A \rangle$ while $E \subset 0 \langle B \rangle$. This implies $\hat{\delta}(C,E) = 1 = \hat{\delta}(D,E)$. Now suppose $\hat{\delta}(C,E) = 1$ and $\hat{\delta}(D,E) = 1$. Then there are sets $A,B \subset P$ such that $\delta(A,B) = 1$ and $C \subset 0 \langle A \rangle$, $E \subset 0 \langle B \rangle$. Also there are sets $A',B' \subset P$ with $\delta(A',B') = 1$ and such that $D \subset 0 \langle A' \rangle$, $E \subset 0 \langle B' \rangle$. Hence we have $E \cap E \subset 0 \langle B \rangle \cap 0 \langle B' \rangle$, and by part 1) of 3.16,

(1) $$E \subset 0 \langle B \cap B' \rangle.$$

Also we have that $C \cup D \subset 0 \langle A \rangle \cup 0 \langle A' \rangle$, and by part 2) of 3.16 this gives:

(2) $$C \cup D \subset 0 \langle A \cup A' \rangle.$$

Since $\delta(A',B') = 1$ and $B \cap B' \subset B'$, we have

(3) $$\delta(A',B \cap B') = 1.$$

Since $\delta(A,B) = 1$ and $B \cap B' \subset B$, then we have

(4) $$\delta(A,B \cap B') = 1.$$

Using property 2) of definition 2.1 for the proximity relation δ, we can combine (3) and (4) to give,

(5) $$\delta(A \cup A', B \cap B') = 1.$$

From (1), (2), and (5) we have $\hat{\delta}(C \cup D,E) = 1$. This concludes the proof of 2).

<u>Verification of 3)</u>. We must prove that two points of uP are close iff they are the same point. We first prove that every point is close to itself. Suppose $\xi \in uP$. On the assumption that $\hat{\delta}(\xi,\xi) = 1$, then there are sets $A, B \subset P$ with $\delta(A,B) = 1$ and such that $\xi \in 0 \langle A \rangle$, $\xi \in 0 \langle B \rangle$. So $A \in \xi$ and $B \in \xi$. Since ξ is an end, then $A \cap B \neq \emptyset$; which contradicts the fact that $\delta(A,B) = 1$. So

we must have $\hat{\delta}(\xi,\xi) = 0$. We now prove the contrapositive of the converse. Suppose $\xi \neq \xi'$, then we want to show $\hat{\delta}(\xi,\xi') = 1$. We now show that it is possible to pick sets $A \in \xi$ and $B \in \xi'$ with $A \cap B = \emptyset$.

<u>Claim 1</u>: If every $A \in \xi$ meets every $B \in \xi'$, then $\xi \cup \xi'$ is a centered δ-system.

<u>Proof of claim 1</u>. To prove $\xi \cup \xi'$ is centered, let $\{D_i \mid i = 1,\cdots,m\}$ be a subsystem of $\xi \cup \xi'$. Say $D_i \in \xi$ for $i = 1,\cdots,k$ and $D_i \in \xi'$ for $i = k+1,\cdots,m$. So we can write:

$$(1) \qquad \bigcap_{i=1}^{m} D_i = \left(\bigcap_{i=1}^{k} D_i\right) \cap \left(\bigcap_{i=k+1}^{m} D_i\right).$$

Let $E = \bigcap_{i=1}^{k} D_i$ and $F = \bigcap_{i=k+1}^{m} D_i$. Then $E \in \xi$ and $F \in \xi'$ because ξ and ξ' are ends.

So we have from (1), and by hypothesis,

$$(2) \qquad \bigcap_{i=1}^{m} D_i = E \cap F \neq \emptyset.$$

(2) proves that $\xi \cup \xi'$ is centered. It is clear that $\xi \cup \xi'$ is a δ-system since if $E \in \xi \cup \xi'$, then E is either in ξ or ξ' hence E is a δ-neighborhood of some set in ξ or ξ'. Thus claim 1 is proven.

Since $\xi \cup \xi'$ is a centered δ-system and $\xi \cup \xi' \supset \xi$ and $\xi \cup \xi' \supset \xi'$, where ξ and ξ' are ends; then $\xi = \xi \cup \xi' = \xi'$. This is a contradiction, since we chose $\xi \neq \xi'$. Hence there must be sets $A \in \xi$ and $B \in \xi'$ with $A \cap B = \emptyset$. But ξ is a δ-system, so there is a set $C \in \xi$ with $A \supset C$. $A \supset C$ means that $\delta(P-A,C) = 1$. Since $A \cap B = \emptyset$, then $B \subset P - A$; so by 2.3 $\delta(B,C) = 1$. $B \in \xi'$, so $\xi' \in 0\langle B\rangle$ and $C \in \xi$, so $\xi \in 0\langle C\rangle$. Thus $\hat{\delta}(\xi,\xi') = 1$. This concludes verification of 3).

<u>Verification of 4)</u>. To prove that every subset of uP is far from the empty set, we will show that uP is far from the empty set. To avoid confusion, we denote the empty set by $\emptyset$ when we consider it as a subset of P and by Λ when we consider it as a subset of uP. Clearly $\Lambda \subset 0\ \langle\emptyset\rangle$. Also uP $\subset 0\ \langle P\rangle$, since every end contains P as an element. $\delta(\emptyset,P) = 1$ follows from part 4) of definition 2.1 for the relation δ. Thus by definition of the relation $\hat{\delta}$, we have $\hat{\delta}(\Lambda,uP) = 1$. This implies that every subset of uP is far from the empty set.

<u>Verification of 5)</u>. Suppose $C,D \subset uP$ and $\hat{\delta}(C,D) = 1$. We must prove the existence of sets $E,F \subset uP$ with $E \cup F = uP$ and $\hat{\delta}(C,E) = 1 = \hat{\delta}(D,F)$. Since $\hat{\delta}(C,D) = 1$, then there are sets $A,B \subset P$ such that $\delta(A,B) = 1$ and $C \subset 0\ \langle A\rangle$, $D \subset 0\ \langle B\rangle$. Since $\delta(A,B) = 1$, then $\delta(P - (P - A),B) = 1$, so that $(P - A) \supset B$. Applying proposition 2.11 5) twice in succession, we get sets $B',B'' \subset P$ with $(P - A) \supset B' \supset B'' \supset B$. Hence we have $\delta(P - B',B'') = 1$ and so $\delta(P - B',P-(P-B'')) = 1$. By proposition 3.16 3), we have that

$$0\ \langle B'\rangle \cup 0\ \langle P - B''\rangle = uP. \tag{1}$$

Since $\delta(A,B') = 1$ and $C \subset 0\ \langle A\rangle$, while $0\ \langle B'\rangle \subset 0\ \langle B'\rangle$,

$$\hat{\delta}(C,0\ \langle B'\rangle) = 1. \tag{2}$$

Since $\delta(P - B'',B) = 1$ and $0\langle P - B''\rangle \subset 0\ \langle P - B''\rangle$, while $D \subset 0\ \langle B\rangle$,

$$\hat{\delta}(D,0\ \langle P - B''\rangle) = 1. \tag{3}$$

From (1), (2), and (3) we see that $0\ \langle B'\rangle$ and $0\ \langle P - B''\rangle$ are the sets E and F in uP which we were looking for; hence property 5) of definition 2.1 is satisfied by $\hat{\delta}$.

Having verified that all five properties of a proximity space are satisfied by uP with the proximity relation $\hat{\delta}$, we have completed the proof of 3.17.

Now we have that uP is indeed a proximity space, and can prove that the proximity space P is embedded in uP.

3.18 *Definition*. We say a proximity space P is *δ-homeomorphically embedded* in the δ-space Q if there is a map $\varphi : P \to Q$ such that φ is 1-1, φ is a δ-map, $\varphi^{-1} : \varphi[P] \to P$ is a δ-map. Such maps as φ are called *δ-homeomorphisms*.

3.19 *Theorem.* If P is any δ-space and uP is the proximity space of all ends of P, then P is δ-homeomorphically embedded in uP.

Proof. We define the function $\psi : P \to uP$, by sending a point $x \in P$ into the end $\xi_x \in uP$. The map is well-defined since we already showed that ξ_x is an end for every $x \in P$ (3.13).

Claim 1: ψ is 1-1.

Proof of claim 1. Suppose $x, y \in P$ and $x \neq y$. By 2.1 3), $\delta(x,y) = 1$. By 2.1 5'), there are disjoint δ-neighborhoods A and B of the sets $\{x\}$ and $\{y\}$. Then $A \in \xi_x$ and $B \in \xi_y$. Since $A \cap B = \emptyset$, then $A \bar{\in} \xi_y$ and $B \bar{\in} \xi_x$; which proves $\xi_x \neq \xi_y$. Hence ψ is 1-1.

Before proving that ψ and ψ^{-1} are δ-maps, we prove the following helpful lemma.

3.20 *Lemma*. For any set $A \subset P$, $\psi^{-1}(0\langle A\rangle) = A^{o}$, the interior of A with respect to the topology $T(\delta)$.

Proof of lemma. Suppose $x \in \psi^{-1}(0\langle A\rangle)$, then $\xi_x \in 0\langle A\rangle$. This means $A \in \xi_x$, and so $A \supset \{x\}$. By lemma 2.12, there exists an open set $U \subset P$ with $A \supset U \supset \{x\}$.

U is an open set containing x and contained entirely in A, so $x \in A^o$. Thus we have,

(1) $$\psi^{-1}(0\langle A\rangle) \subset A^o.$$

If $x \in A^o$, then $x \bar{\in} P - A^o$. A^o is open, so $P - A^o$ is closed, from which it follows that $\delta(x,P - A^o) = 1$. Hence we have $A^o \supset \{x\}$. But $A \supset A^o$, so $A \supset \{x\}$. Thus $A \in \xi_x$ and $\xi_x \in 0\langle A\rangle$ and $x \in \psi^{-1}(0\langle A\rangle)$. So we have shown that,

(2) $$A^o \subset \psi^{-1}(0\langle A\rangle).$$

(1) and (2) prove the lemma.

We now return to the proof of theorem 3.19. We still have to prove that the maps ψ and ψ^{-1} are δ-maps.

<u>Claim 2</u>: ψ is a δ-map.

<u>Proof of claim 2.</u> Let $C,D \subset uP$ with $\hat{\delta}(C,D) = 1$. Then there exist sets $A,B \subset P$ with $\delta(A,B) = 1$, and $C \subset 0\langle A\rangle$, while $D \subset 0\langle B\rangle$. Thus we have,

(3) $$\psi^{-1}(C) \subset \psi^{-1}(0\langle A\rangle),$$

and

(4) $$\psi^{-1}(D) \subset \psi^{-1}(0\langle B\rangle).$$

Applying lemma 3.20 to (3) and (4) we have,

(5) $$\psi^{-1}(C) \subset A^o,$$

and

(6) $$\psi^{-1}(D) \subset B^o.$$

Since $\delta(A,B) = 1$ and $A^o \subset A$ and $B^o \subset B$, then we have,

(7) $$\delta(A^o,B^o) = 1.$$

Applying proposition 2.3 to (7) together with (5) and (6), we have,

(8) $$\delta(\psi^{-1}(0), \psi^{-1}(D)) = 1.$$

From (8) and proposition 2.20, we may conclude that ψ is an δ-map.

<u>Claim 3</u>: ψ^{-1} is a δ-map.

<u>Proof of claim 3</u>. Since we already have that ψ is 1-1, then $(\psi^{-1})^{-1} = \psi$. So if we apply proposition 2.20 to prove ψ^{-1} is a δ-map, we must only show that whenever $\delta(A,B) = 1$, then $\hat{\delta}(\psi(A), \psi(B)) = 1$. If $A,B \subset P$ and $\delta(A,B) = 1$, then also $\delta(P - (P - A),B) = 1$; so that $P - A \supset B$. By 2.11 5), there are sets $C,D \subset P$ with $P - A \supset C \supset D \supset B$. Since by lemma 2.12 there is an open set U such that $D \supset U \supset B$, and since D^o is the largest open set contained in D, then $D^o \supset B$. Of course $D^o \supset B$ implies $D^o \supset B$. By lemma 3.20, $D^o = \psi^{-1}(0 \langle D \rangle)$, so:

(9) $$\psi^{-1}(0 \langle D \rangle) \supset B.$$

Every map preserves set inclusion, so from (9) we have,

(10) $$\psi[\psi^{-1}(0 \langle D \rangle)] = \psi(B).$$

Since $\psi \psi^{-1} = 1$ uP, then (10) gives us,

(11) $$0 \langle D \rangle \supset \psi(B).$$

Since $P - A \supset C$, then we also have $P - C \supset A$. As we noted previously, this means that $(P - C)^o \supset A$ and hence that $(P - C)^o \supset A$. Applying lemma 3.20, we get

$\psi^{-1}(0 \langle P - C\rangle) \supset A$, from which it follows that,

(12) $$0 \langle P - C\rangle \supset \psi(A).$$

Since $\delta(P - C,D) = 1$, then

(13) $$\hat{\delta}(0 \langle D\rangle, 0 \langle P - C\rangle) = 1.$$

From (11), (12), and (13) it follows that

(14) $$\hat{\delta}(\psi(A), \psi(B)) = 1.$$

Hence the proof of claim 3 is complete.

Claims 1,2, and 3 prove that P is δ-homeomorphically embedded in uP, so that theorem 3.19 is true.

Since δ-homeomorphisms are homeomorphisms (2.21), we can now treat P as a subspace of uP simply by indentifying each point $x \in P$ with the end $\xi_x \in uP$.

Having made this identification, we can restate lemma 3.20 in the form:

3.21 $\quad P \cap 0 \langle A\rangle = A^o.$

We have finally arrived at the place where we can prove that the construction of uP from P was worthwhile, because uP is exactly what we wanted — a δ-extension of P.

3.22 <u>Theorem</u>. The δ-space uP is a δ-extension of the δ-space P.

<u>Proof</u>. Since we already have that P is a subset of uP, and since P being δ-homeomorphically embedded in uP insures that the original proximity in P is the same as that induced on P as a subset of uP; then all we have left to do is show

that P is a dense subset of uP. To do this, we show that every point of uP is close to P. Suppose $\xi \in uP$. If $\hat{\delta}(\xi,P) = 1$, then there are sets $A,B \subset P$ with $\delta(A,B) = 1$ and $\xi \in 0\langle A\rangle$, while $P \subset 0\langle B\rangle$. Since $P = \{\xi_y | y \in P\} \subset 0\langle B\rangle$, this means $B \in \xi_y$ for every $y \in P$. But then $y \in B$ for every $y \in P$, which proves $B = P$. Since $\delta(A,B) = 1$, then $A = \emptyset$. Since $\xi \in 0\langle A\rangle$, then $A = \emptyset$ is a contradiction. Thus we have $\hat{\delta}(\xi,P) = 0$, and theorem 3.22 is proven.

<u>The construction of uP is now complete</u>.

Not only is uP a δ-extension of P, but we shall prove that uP is an absolutely closed δ-space. This will be important since we will then prove the converse of proposition 3.3; that is, every absolutely closed δ-space is compact. When all this is done we will have P as a subspace of the compact space uP, and thus we will have proved P to be completely regular.

3.23 <u>Theorem</u>. A δ-space P is absolutely closed if and only if every centered δ-system of sets from P has non-empty intersection; that is, iff every centered δ-system has the finite intersection property.

<u>Proof</u>. We first prove the contrapositive of the "if" part of 3.23. So we suppose P is not absolutely closed, and try to find some centered δ-system of P whose intersection is empty. P is not absolutely closed is equivalent to saying there is a point $y \mathrel{\bar{\in}} P$ and $P \cup \{y\} = P'$ is a δ-space containing P as a dense subspace. Let ξ_y denote the end of P' consisting of all δ-neighborhoods of $\{y\}$. Let $\xi = \xi_y' \cap \{P\} = \{E \cap P | E \in \xi_y\}$.

<u>Claim 1</u>: ξ is centered.

<u>Proof of claim 1</u>. Suppose $\{E_i | i = 1,\cdots,k\}$ is a subsystem of ξ. Then for each i, there is an $H_i \subset P'$ with $E_i = H_i \cap P$ and $H_i \in \xi_y$. So we have:

(1) $$\bigcap_{i=1}^{n} E_i = \bigcap_{i=1}^{n} (H_i \cap P) = (\bigcap_{i=1}^{n} H_i) \cap P.$$

Since ξ_y is an end, then $\bigcap_{i=1}^{n} H_i \in \xi_y$. If $\bigcap_{i=1}^{n} H_i = \{y\}$, then $\{y\} \supset \{y\}$. So we have $\hat{\delta}(P' - \{y\},y) = 1$, which means

(2) $$\hat{\delta}(P,y) = 1.$$

(2) is impossible since P is dense in P'. Thus there must be an element $x \neq y$ and $x \in \bigcap_{i=1}^{n} H_i$. Since $x \neq y$, then $x \in (\bigcap_{i=1}^{n} H_i) \cap P$. So from (1) we have that $\bigcap E_i \neq \emptyset$. Hence claim 1 is proven.

<u>Claim 2</u>: ξ is a δ-system in P.

<u>Proof of claim 2.</u> Suppose $A \in \xi$. Then $A = H \cap P$ for some $H \in \xi_y$. Since ξ_y is a δ-system, then there is an $H' \in \xi_y$ with $H \supseteq H'$. Let $A' = H' \cap P$. Then $A \supseteq A' \in \xi$. This completes the proof of claim 2.

<u>Claim 3</u>: $\bigcap \{A | A \in \xi\} = \emptyset$.

<u>Proof of claim 3.</u> Since every $A \in \xi$ is of the form $H \cap P$, where $H \in \xi_y$, we can write:

(3) $$\bigcap \{A | A \in \xi\} = \bigcap \{H \cap P | H \in \xi_y\},$$

and using commutativity of intersection in (3) we have,

(4) $$\bigcap \{A | A \in \xi\} = \bigcap \{H | H \in \xi_y\} \cap P.$$

Thus in order to prove claim 3 it is sufficient to prove that

(5) $$\bigcap \{H | H \in \xi_y\} = \{y\}.$$

If there were an $x \in \bigcap\{H|H \in \xi_y\}$ and $x \neq y$, then $\delta(x,y) = 1$. So there would be disjoint δ-neighborhoods C of $\{x\}$ and D of $\{y\}$, so $x \in D$. $x \in D$ contradicts the fact that C and D are disjoint. Thus (5) is verified, and the proof of claim 3 is complete.

Claims 1, 2, and 3 prove that ξ is a centered δ-system with empty intersection.

We now prove the contrapositive of the "only if" part of 3.23. Suppose ξ' is a centered δ-system in P with empty intersection. By proposition 3.11, ξ' is contained in some end ξ of P. ξ must have empty intersection because it contains the system ξ', which has empty intersection. Since every end of the form ξ_x, where $x \in P$, has non-empty intersection; then ξ is not one of these ends. Identifying the points $x \in P$ with the ends $\xi_x \in uP$, we have that uP contains P properly, because $\xi \in uP$ and $\xi \notin P$. This proves P is not absolutely closed.

We have thus completed the proof of theorem 3.23.

3.24 Theorem. For any δ-space P, the δ-extension uP is absolutely closed.

Proof. According to theorem 3.23, we must show that every centered δ-system in uP has a non-empty intersection. Let η' be an arbitrarily chosen centered δ-system in uP. Extend η' via proposition 3.11 to an end η of the space uP. We now define $\xi' = \{A \subset P|A = H \cap P \text{ for some } H \in \eta\}$. Following a procedure like that in claims 1 and 2 of theorem 3.23, we arrive at ξ' being a centered δ-system in P. Again we invoke proposition 3.11 to extend ξ' to an end ξ of the space P, that is $\xi \in uP$. We now show:

(1) $$\xi \in \bigcap\{H \subset uP|H \in \eta'\}.$$

If (1) were not true, then there would be an $H \in \eta'$ such that $\xi \mathrel{\bar{\in}} H$. η' is a δ-system, so there is a set $H' \in \eta'$ with $H \supset H'$. Thus we have $\hat{\delta}(uP - H, H') = 1$, where $\hat{\delta}$ is the proximity relation in uP. $\xi \in uP - H$, so we also have,

(2) $$\hat{\delta}(\xi, H') = 1.$$

From (2) and definition 3.15, follows the existence of sets $A, B \subset P$ with:

(3) $$\delta(A,B) = 1,$$

and

(4) $$\xi \in 0\langle A\rangle,$$

while

(5) $$H' \subset 0\langle B\rangle.$$

From (5), 5.20, and 5.21, it follows that,

(6) $$P \cap H' \subset P \cap 0\langle B\rangle = B^{o} \subset B.$$

From (3) and (6), we have:

(7) $$\delta(A, P \cap H') = 1.$$

(7) clearly implies that,

(8) $$A \cap (P \cap H') = \emptyset.$$

Since $H' \in \eta'$, then $P \cap H' \in \xi$; and (4) implies that $A \in \xi$. So (8) contradicts the fact that ξ is an end. We have proven that (1) holds, thus completing the proof of theorem 3.24.

3.25 Theorem. A proximity space is absolutely closed iff it is compact.

Proof. "If" was proven in proposition 3.3. For "only if" we use the characterization of compactness in theorem 1.2. So we are trying to prove that every centered system of closed sets in an absolutely closed δ-space P has non-empty intersection. Let Φ be any centered system of closed sets in P. For each $\varphi \in \Phi$, let ξ_φ be the centered δ-system in P consisting of all the δ-neighborhoods of φ. Let $\xi = \bigcup\{\xi_\varphi | \varphi \in \Phi\}$. We shall prove that ξ is a centered δ-system having non-empty intersection. ξ is a δ-system: If $E \in \xi$, then $E \in \xi_\varphi$ for some $\varphi \in \Phi$, and ξ_φ is a δ-system, so there is a $D \in \xi_\varphi$ hence $D \in \xi$ with $E \supseteq D$. ξ is centered: If $\{E_i | i = 1,\cdots,n\}$ is a finite subsystem of ξ, then for each E_i there is a $\varphi_i \in \Phi$ with $E_i \in \xi_{\varphi_i}$. But Φ is centered, so $\bigcap_{i=1}^n E_i \supset \bigcap_{i=1}^n \varphi_i \neq \emptyset$. Since P is absolutely closed, then theorem 3.23 verifies that ξ has non-empty intersection. That is, we have that,

$$(1) \qquad \bigcap\{H \subset P | H \in \xi\} \neq \emptyset.$$

From the definition of ξ, we have:

$$(2) \qquad \bigcap\{H|H \in \xi\} = \bigcap\{H|H \in (\bigcup_{\varphi\in\Phi} \xi_\varphi)\}.$$

Restating the right hand side of (2) we have:

$$(3) \qquad \bigcap\{H|H \in \xi\} = \bigcap_{\varphi\in\Phi} (\bigcap\{H|H \in \xi_\varphi\}) .$$

Employing lemma 2.13 for each $\varphi \in \Phi$ on the right hand side of (3) gives us:

$$(4) \qquad \bigcap\{H|H \in \xi\} = \bigcap_{\varphi\in\Phi} \bar{\varphi} = \bigcap\{\bar{\varphi}|\varphi \in \Phi\}.$$

Since Φ is a system of closed sets, we can replace $\bar{\varphi}$ by φ in the right hand side of (4) and have:

(5) $$\bigcap\{H \mid H \in \xi\} = \bigcap\{\varphi \mid \varphi \in \Phi\}.$$

From (5) and (1), we conclude that Φ has non-empty intersection, which completes the proof of theorem 3.25.

3.26 Corollary. Every δ-space P, considered as a topological space, is completely regular.

Proof. If P is a δ-space, then by theorem 3.25 uP is an absolutely closed δ-extension of P. By theorem 3.25, uP, being absolutely closed, is compact. uP is compact implies uP is normal. Since every subspace of a normal space is completely regular, and since P is a subspace of the normal space uP, then P is completely regular.

We close the chapter with a theorem which strengthens theorem 3.24. After proving this theorem, we will have uP as the unique absolutely closed δ-extension of P.

3.27 Theorem. Every δ-space P has only one absolutely closed δ-extension, up to δ-homeomorphism.

Proof. Suppose P has an absolutely closed δ-extension vP different from uP. That is, uP and vP are both absolutely closed δ-extensions of P, and there does not exist a surjective δ-homeomorphism $f\colon vP \to uP$ with $f_{|P} = 1_P$. By theorem 3.25, vP and uP are compactifications of P when considered as topological spaces. Since every continuous map from a compact δ-space into any other δ-space is also a δ-map (2.22), then there is no homeomorphism from vP onto uP which is the identity on points of P. From Theorem 1.12, we conclude that there must be sets $A,B \subset P$ such that either $cl_{uP}(A) \cap cl_{uP}(B) = \emptyset$ whereas $cl_{uP}(A) \cap cl_{vP}(B) \neq \emptyset$, or $cl_{uP}(A) \cap cl_{uP}(B) \neq \emptyset$ whereas $cl_{vP}(A) \cap cl_{vP}(B) = \emptyset$. This is an impossible situation, since uP and vP must induce the same proximity relation on P. Thus uP is the unique absolutely closed δ-extension of P.

IV. Relationship between Proximity Spaces and Compactifications

In Chapter III, we have seen that for a given completely regular space X, each proximity space (X,δ) consistent with it gives rise to a compactification of X, and conversely. In this chapter, our immediate goal is to show that this assignment defines not only a 1-1 correspondence, but an "order" preserving isomorphism between the collection of proximity spaces consistent with X and the class of compactifications of X. Moreover, in the process of proving this isomorphism, we uncover some nice properties of δ-spaces.

4.1 Theorem. Let X be a completely regular space, let

$$\mathcal{R} = \{(X,\delta_\alpha) | \alpha \in \mathcal{A}\}$$

be the collection of all δ-spaces consistent with X, and let $\mathcal{B}$ be the collection of all compactifications of X. Then there is a 1-1 correspondence between $\mathcal{R}$ and $\mathcal{B}$.

Proof. We define a map $\varphi: \mathcal{R} \to \mathcal{B}$ as follows: if $(X,\delta_\alpha) \in \mathcal{R}$, let $\varphi(X,\delta_\alpha) = u_\alpha X$, the unique absolutely δ-extension of (X,δ_α) [see 3.27], which is also compact by 3.25. Then φ is well-defined and is 1-1, since each proximity relation δ_α on X gives rise to a unique compactification $u_\alpha X$ which, in turn, induces a unique proximity on $u_\alpha X$ whose restriction to X is δ_α. Finally, since each compactification Y of X is associated with a unique proximity space (Y,δ) consistent with it, then $(X,\delta|_{\mathcal{P}X})$ is a proximity space consistent with X and $\varphi(X,\delta|_{\mathcal{P}X}) =$ Y. Hence φ is onto. 4.1 is proved.

As we have observed in Chapter I, there is a natural partial ordering $\geq$ in $\mathcal{B}$. This ordering should induce a partial ordering in $\mathcal{R}$, by using the map φ defined in 4.1. In fact, this ordering will turn out to be just as natural as the one in $\mathcal{B}$.

4.2 Definition. If (X,δ_α) and (X,δ_γ) are δ-spaces, we define $(X,\delta_\alpha) \gg (X,\delta_\gamma)$ if the identity map

$$1_X\colon (X,\delta_\alpha) \to (X,\delta_\gamma)$$

is a δ-map.

4.3 Proposition. The relation "$\gg$" defined in 4.2 is a partial ordering for the set $\mathcal{R}$ of all δ-spaces (X,δ_α) consistent with the space X.

Proof. Reflexivity and anti-symmetry are clear, and transitivity of the relation "$\gg$" follows from the fact that the composition of two δ-maps is again a δ-map.

Since the ordering $\gg$ is defined in terms of δ-maps whereas the ordering $\geq$ is defined in terms of continuous functions, we need a connection between δ-maps defined on δ-spaces and continuous functions defined on the associated compact spaces.

4.4 Theorem. Every δ-map $f\colon (X,\delta) \to (Y,\delta)$ can be extended to a continuous map, and therefore a δ-map, from the δ-extension uX of (X,δ) to the δ-extension uY of (Y,δ').

Proof. By theorem 1.11, we need only show that if A and B are disjoint closed sets in uY, then $cl_{uX}(f^{-1}[A])$ and $cl_{uX}(f^{-1}[B])$ are disjoint. Let A and B be disjoint closed sets in uY. Then A and B are far in the (induced) proximity of uY. Since f is a δ-map into uY, then $\delta(f^{-1}[A], f^{-1}[B]) = 1$. Since the proximity δ is that induced by the compactification uX, then $cl_{uX}(f^{-1}[A])$ and $cl_{uX}(f^{-1}[B])$ are disjoint, as desired.

4.5 Theorem. For any completely regular space X, the bijective map $\varphi: \mathfrak{R} \to \mathfrak{B}$ of 4.1, is an isomorphism relative to the partial orderings $\gg$ and $\geq$ of $\mathfrak{R}$ and $\mathfrak{B}$, respectively.

Proof. Suppose $(X,\delta_\alpha) \gg (X,\delta_\gamma)$ in $\mathfrak{R}$. Then the map $1_X: (X,\delta_\alpha) \to (X,\delta_\gamma)$ is a δ-map, and therefore can be extended to a continuous map from the δ-extension $u_\alpha X$ of (X,δ_α) into the δ-extension $u_\gamma X$ of (X,δ_γ), by theorem 4.4. Thus $u_\alpha X \geq u_\gamma X$ in $\mathfrak{B}$. Since $\varphi(X,\delta_\alpha) = u_\alpha X$ and $\varphi(X,\delta_\gamma) = u_\gamma X$ (see definition of φ in the proof of 4.1), we have $\varphi(X,\delta_\alpha) \geq \varphi(X,\delta_\gamma)$.

Suppose now that $Y,Z \in \mathfrak{B}$ with $Y \geq Z$. Since φ is onto, there are (X,δ_α), $(X,\delta_\gamma) \in \mathfrak{R}$ such that $\varphi(X,\delta_\alpha) = Y \geq Z = \varphi(X,\delta_\gamma)$. Hence, by definition of $\geq$, there is a continuous $f: \varphi(X,\delta_\alpha) = Y \to Z = \varphi(X,\delta_\gamma)$ such that $f|_X = 1_X$. By 2.22, f is also a δ-map so that its restriction to X, which is 1_X, is a δ-map from (X,δ_α) into $X,\delta_\gamma)$. Thus we have $\varphi^{-1}(Y) = (X,\delta_\alpha) \geq (X,\delta_\gamma) = \varphi^{-1}(X)$.

Hence φ is an isomorphism.

We have thus successfully represented the compactifications of a given completely regular space X by δ-spaces consistent with the space X, and therefore problems of compactifications are translated to appropriate problems of δ-spaces. Due to the relatively simple structure of δ-spaces, we have the hope of solving some of these problems.

Before we go into the solutions of these problems (see Chapter V), we wish to point out other niceties of δ-spaces by proving the following analogues of the Urysohn and Tietze theorems.

4.6 Theorem. Let (P,δ) be a proximity space. $A,B \subset P$ are far apart if and only if they are separated by a δ-map, i.e., iff there is a δ-map $f: P \to [0,1]$ such that $f[A] = 0$ and $f[B] = 1$.

Proof. Suppose $f: P \to [0,1]$ is a δ-map such that $f[A] = 0$, $f[B] = 1$. Since $\{0\}$ and $\{1\}$ are disjoint closed subsets of the compact space $[0,1]$, so if $\hat{\delta}$ denotes the unique proximity relation on $[0,1]$, we have $\hat{\delta}(0,1) = 1$. Since f is a δ-map, then $\delta(f^{-1}(0), f^{-1}(1)) = 1$, from which it follows that $\delta(A,B) = 1$. Conversely, suppose $A,B \subset P$ with $\delta(A,B) = 1$, then $cl_{uP}(A) \cap cl_{uP}(B) = \emptyset$, where uP denotes the unique absolutely closed δ-extension of (P,δ). Since uP is compact, in particular, uP is normal, there is a continuous function $F: uP \to [0,1]$ such that $F[cl_{uP}(A)] = 0$, $F[cl_{uP}(B)] = 1$. Since F is defined on a compact space and is continuous, F is a δ-map, by 2.21. Since $F[A] = 0$, $F[B] = 1$, and $A,B \subset P$, then $F|_P: P \to [0,1]$ is a δ-map which separates A and B. The proof of 4.8 is complete.

4.7 Theorem. Every bounded real-valued δ-map f defined on a subset A of the δ-space P and satisfying $|f| \leq M$, can be extended to a δ-map $F: P \to R$ satisfying $|F| \leq M$.

Proof. Since $A \subset P$, then $cl_{uP}(A)$ is a compact subspace of the absolutely closed (compact) δ-extension uP of the δ-space P. It follows from the uniqueness of δ-extensions that $cl_{uP}(A)$ is the δ-extension of $(A,\delta|_A)$ and that $[-M,M]$ is the δ-extension of itself considered as a δ-space. By theorem 4.4, f can be extended to a continuous map $F': cl_{uP}(A) \to [-M,M]$. Now F' is a continuous map from the closed subspace $cl_{uP}(A)$ of the compact, in particular, normal, space uP; so, by Tietze's characterization of normality, F' can be extended to a continuous $f'': uP \to [-M,M]$. Since F'' is continuous on a compact space, F'' is a δ-map. Let $F = F''|_P$, then F is the required extension of f. The theorem is proved.

Finally, in this chapter, we list some properties of the operator $0 < >$, which are needed to prove a lemma that becomes useful in applying the theory of

δ-spaces to the study of compactifications (see Chapter V).

4.8 If (P,δ) is a proximity space, then:

1) for any $A \subset P$, $0\langle Int_P A\rangle = 0\langle A\rangle$;

2) for any $A \subset P$, $0\langle A\rangle$ is open in uP, the δ-extension of (P,δ);

3) the set

$$\{0\langle U\rangle \mid U \text{ is an open set in } P\}$$

forms a basis for the topology of uP.

Proof. 1). Since $Int_P A \subset A$, we have $0\langle Int_P A\rangle \subset 0\langle A\rangle$, by 2) of 3.16. To prove the reverse inclusion, let $\xi \in 0\langle A\rangle$. Then $A \in \xi$ so that A is a δ-neighborhood of some $B \in \xi$. By 2.12, $B \subset Int_P A$, so that $Int_P A \in \xi$. Hence $\xi \in 0\langle Int_P A\rangle$. 1) is proved.

2). By 2.12, it suffices to show that $0\langle A\rangle$ is a δ-neighborhood of each end $\xi \in 0\langle A\rangle$. Let $\xi \in 0\langle A\rangle$. Then $A \in \xi$, so that there are sets B and C in ξ such that $A \supset B \supset C$. Since B and $P - A$ are far apart $uP = 0\langle P - B\rangle \cup 0\langle A\rangle$, by (3) of 3.16, so that $uP - 0\langle A\rangle \subset 0\langle P - B\rangle$. But $\xi \in 0\langle C\rangle$, and the sets C and $P - B$ are far apart; it follows from our definition of the induced proximity $\hat{\delta}$ on uP that $\hat{\delta}(\xi, uP - 0\langle A\rangle) = 1$; i.e., $0\langle A\rangle$ is a δ-neighborhood of ξ, as desired. Thus 2) is proved.

3). Let $\xi \in H$ with H open in uP. Since $uP - H$ is closed, $\hat{\delta}(\xi, uP - H) = 1$, where $\hat{\delta}$ is the induced proximity on uP. By the definition of $\hat{\delta}$, there are sets A and B far apart in P such that $\xi \in 0\langle A\rangle$ and $uP - H \subset 0\langle B\rangle$. Hence $\hat{\delta}(0\langle A\rangle, uP - H) = 1$; i.e., $0\langle A\rangle \subset H$. But $0\langle Int_P A\rangle = 0\langle A\rangle$ by 1), and $Int_P A$ is open in P. Hence 3) is proven.

4.9 Lemma. For any set A of a given δ-space (P,δ), the set $0\langle A\rangle$ is the largest of all open sets H of the δ-extension uP of (P,δ) such that $H \cap P = \mathrm{Int}_P A$.

<u>Proof</u>. Let U be the largest of all open sets H of uP such that $H \cap P = \mathrm{Int}_P A$. By 4.8 3),

$$U = \bigcup_{\lambda \in \Lambda} 0 \langle U_\lambda \rangle$$

for some family $\{U_\lambda \mid \lambda \in \Lambda\}$ of open sets in P. By 2) of 3.16,

$$(\alpha) \qquad U = \bigcup_{\lambda \in \Lambda} 0 \langle U_\lambda \rangle \subset 0 \langle \bigcup_{\lambda \in \Lambda} U_\lambda \rangle.$$

By definition of U and 3.21,

$$\mathrm{Int}_P A = P \cap U = P \cap \left(\bigcup_{\lambda \in \Lambda} 0 \langle U_\lambda \rangle \right) = \bigcup_{\lambda \in \Lambda} U_\lambda .$$

It now follows from (α) and 4.7 1) that

$$0 \langle A \rangle = 0 \langle \mathrm{Int}_P A \rangle = 0 \langle \bigcup_{\lambda \in \Lambda} U_\lambda \rangle \supset U.$$

On the other hand, it follows from 4.7 2) and the definition of U that

$$U \supset 0 \langle \mathrm{Int}_P A \rangle = 0 \langle A \rangle.$$

Hence $0 \langle A \rangle = U$, as was to be proved.

V. Applications

In the preceding chapters we went to a great deal of trouble defining and proving certain properties of proximity spaces. Hopefully our work was not in vain; and to prove that it wasn't, we demonstrate the usefulness of the theory.

We need not go far to find a use for δ-spaces; in fact, we already noted that the category of all proximity spaces coincides exactly with the category of completely regular spaces, so that we can use the theory of δ-spaces in the study of completely regular spaces. An important use of the theory was made by V. A. Efrimovic when he characterized uniformly continuous functions in metric spaces as the δ-maps in the δ-spaces associated with these metric spaces.

In this chapter, we study another application of the theory of δ-spaces, as had been done originally by E. G. Skljarenko. Our probes reach into the theory of compactifications, with an ultimate aim (in this paper at least) of producing an extension theorem, and a proof of the Freudenthal-Morita theorem (i.e., a sufficient condition for a space to have a compactification with zero-dimensional annex and with weight the same as that of the original space). These results enabled Skljarenko to solve, in the negative, an outstanding problem of P. S. Aleksandrov. Namely, is it true that every peripherally compact space has a compactification (with a zero-dimensional annex) of the same dimensionality as that of the space itself?

It is known that any homeomorphism between completely regular spaces X_1 and X_2 can be extended to a homeomorphism between the Stone-Čech compactifications X_1 and X_2. In this paper we do not relax the prerequisites on the function, but we do show that one is not forced to choose the Čech compactification to retain the force of the extension. This improved extension theorem is based on the concept of a perfect compactification.

5.1 Definition. A compactification Y of the completely regular space X is said to be perfect with respect to the open set $U \subset X$ if:

(1) $$\mathrm{Fr}_Y 0 \langle U \rangle = \mathrm{cl}_Y(\mathrm{Fr}_X U).$$

Y is a perfect compactification of X if it is perfect with respect to every open set $U \subset X$.

This definition immediately reminds us that the set, $\{\alpha X\}$, of all compactifications of a given completely regular space X, is isomorphic to the set, $\{X_\alpha\}$, of all proximity spaces consistent with the space X. We are simultaneously envigorated by the hope of discovering which, if any, of the δ-spaces X_α correspond to perfect compactifications of the space X.

5.2 Lemma. Let X be a completely regular space, δ a proximity on X consistent with the given topology on X, and let Y a compactification corresponding to δ. Then Y is perfect with respect to the open set $U \subset X$ if and only if for every set $A \subset U$, $\delta(A,\mathrm{Fr}_X U) = 1$ implies $\delta(A, X - U) = 1$.

Proof. Let Y be a compactification which is perfect with respect to the open set $U \subset X$, and A be contained in U such that $\delta(A,\mathrm{Fr}_X U) = 1$. Thus we have:

(1) $$\mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(\mathrm{Fr}_X U) = \emptyset.$$

To show $\delta(A, X - U) = 1$, we assume that $\delta(A, X - U) = 0$, i.e., we assume:

(2) $$\mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(X - U) \neq \emptyset,$$

and we search for a contradiction. If (2) holds, then there is a point $\xi \in [\mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(X - U)]$. Since $A \subset U \subset 0 \langle U \rangle$, then we have:

(3) $$\mathrm{cl}_Y(A) \subset \mathrm{cl}_Y(0\langle U\rangle).$$

From (3) we have $\xi \in \mathrm{cl}_Y(0\langle U\rangle)$. Since $0\langle U\rangle = Y - \mathrm{cl}_Y(X - U)$ and $\xi \in \mathrm{cl}_Y(X - U)$, then $\xi \notin 0\langle U\rangle$. Thus $\xi \in \mathrm{Fr}_Y 0\langle U\rangle$, but from (1) we see that $\xi \notin \mathrm{cl}_Y(\mathrm{Fr}_X U)$; hence we have a contradiction to the fact that Y is perfect with respect to the set U. We now let the condition of the lemma be fulfilled for the open set $U \subset X$, and show that

(4) $$\mathrm{cl}_Y(\mathrm{Fr}_X U) = \mathrm{Fr}_Y 0\langle U\rangle.$$

Hence we have:

(5) $$\mathrm{Fr}_X U \subset \mathrm{Fr}_Y 0\langle U\rangle.$$

From (5) it follows:

(6) $$\mathrm{cl}_Y(\mathrm{Fr}_X U) \subset \mathrm{Fr}_Y 0\langle U\rangle.$$

Having (6), we need the following lemma to prove (4).

5.3 Lemma. Let X be a completely regular space with compactification Y, and proximity relation δ on X corresponding to Y. If V' and V'' are open subsets of X with $\delta(V',V'') = 1$, then:

(1') $$0\langle V' \cup V''\rangle = 0\langle V'\rangle \cup 0\langle V''\rangle.$$

Proof. We already have:

(2') $$0\langle V' \cup V''\rangle \supset 0\langle V'\rangle \cup 0\langle V''\rangle,$$

so we prove the reverse inclusion. Let $V = V' \cup V''$, and $\xi \in 0\langle V\rangle$. Since V is dense in $0\langle V\rangle$, then

(3') $$\xi \in cl_Y(V) = [cl_Y(V') \cup cl_Y(V'')].$$

Since $\delta(V', V'') = 1$, then

(4') $$cl_Y(V') \cap cl_Y(V'') = \emptyset.$$

From (3') and (4'), either $\xi \in cl_Y(V')$ and $\xi \notin cl_Y(V'')$, or $\xi \in cl_Y(V'')$ and $\xi \notin cl_Y(V')$. We verify the reverse inclusion from (2') only in the second case, since the two cases are analogous. Since Y is compact, then we can find a neighborhood U' of the point ξ, with $U' \cap cl_Y(V') = \emptyset$. Let $U = U' \cap 0\langle V\rangle$, so that U is open in Y, U contains ξ, and $U \cap cl_Y(V') = \emptyset$. We now have:

(5') $$U \cap X \subset 0\langle V\rangle \cap X = V = V' \cup V''.$$

Since $U \cap cl_Y(V') = \emptyset$, then we must have:

(6') $$U \cap X \subset V'' = 0\langle V''\rangle \cap X.$$

So $\xi \in U$ implies $\xi \in 0\langle V''\rangle$. Thus, $0\langle V' \cup V''\rangle = 0\langle V'\rangle \cup 0\langle V''\rangle$; and 5.3 is proved.

<u>Proof of 5.2</u> (continued). We have yet to show:

(7) $$Fr_Y 0\langle U\rangle \subset cl_Y(Fr_X U).$$

Let $\xi \notin cl_Y(Fr_X U)$. Since Y is compact, we can choose a neighborhood W of ξ such that:

(8) $$cl_Y(W) \cap cl_Y(Fr_X U) = \emptyset.$$

Let $V' = W \cap U$ and $V'' = W \cap [X - cl_X(U)]$. From our definition of V' and V'' we get:

(9) $$W \cap X = V' \cup V''.$$

Since $0 \langle V' \cup V'' \rangle$ is the largest open set in Y exciseing $V' \cup V''$ from X (4.9), it follows from (9) that

(10) $$\xi \in 0 \langle V' \cup V'' \rangle.$$

From (8) and the definition of V', we have $\delta(V', Fr_X U) = 1$. By the condition of lemma 5.2, we also have $\delta(V', X - U) = 1$, from which it follows that $\delta(V', V'') = 1$. We now invoke lemma 5.3 in (10), so that either $\xi \in 0 \langle V' \rangle$ or $\xi \in 0 \langle V'' \rangle$. If $\xi \in 0 \langle V' \rangle$, then $\xi \in 0 \langle U \rangle$, so $\xi \notin Fr_Y 0 \langle U \rangle$. If $\xi \in 0 \langle V'' \rangle$, then $\xi \in 0 \langle X - cl_X(U) \rangle$. Since

$$0 \langle U \rangle \cap 0 \langle X - cl_X(U) \rangle = 0 \langle U \cap [X - cl_X(U)] \rangle = 0 \langle \emptyset \rangle = \emptyset,$$

then $0 \langle X - cl_X(U) \rangle$ is a neighborhood of ξ not meeting $0 \langle U \rangle$; which proves $\xi \notin Fr_Y\, 0 \langle U \rangle$. Thus, if $\xi \notin cl_Y(Fr_X U)$, then $\xi \notin Fr_Y 0 \langle U \rangle$. This completes the proof of lemma 5.2.

5.4 Corollary. The Stone-Cech compactification βX of a completely regular space X is a perfect compactification.

Proof. Let δ be the proximity relation on X corresponding to βX. Let U be an arbitrary open set in X, and $A \subset U$ with $\delta(A, Fr_X U) = 1$. By (4.6), there is a continuous function $f: X \to [0,1]$ such that $f(a) = 0$ for every $a \in A$, $f(x) = 0$ for every $x \in Fr_X U$, and $0 \leq f(x) \leq 1$, for every $x \in X$. We now define a function $g: X \to [0,1]$ as follows:

$$g(x) = \begin{cases} f(x), & \text{if } x \in cl_X(U); \text{ and} \\ 1, & \text{if } x \in X - U. \end{cases}$$

Since $f = 1$ on $Fr_X U$; then g is well-defined. It now follows that g is continuous and separates the sets A and X - U, so $\delta(A, X - U) = 1$ (4.6). From 5.2, it follows that 5.4 is proved.

We found in corollary 5.4 that lemma 5.2 made it easy to prove that the Čech compactification is perfect. However, if we try to prove that some other compactifications are perfect, we find the representation given in 5.2 hard to handle. So we prove several other characterizations of a perfect compactification.

5.5 Definition. We say a closed set $F \subset X$ separates (or splits) the space X into sets U_1 and U_2, if $X - F = U_1 \cup U_2$, where U_1 and U_2 are disjoint open sets in X.

5.6 Definition. The set $N \subset Y$ separates (or splits) the space Y at the point $x \in N$, if the point x has a neighborhood U in Y such that

$$U \cap (Y - N) = V' \cup V'',$$

where V' and V'' are disjoint open sets in $Y - N$, and $x \in [cl_Y(V') \cap cl_Y(V'')]$.

5.7 Theorem. Let Y be a compactification of the completely regular space X. The following properties of the compactification Y are equivalent:

1) Y is a perfect compactification of X.

2) The annex, Y - X, does not split the compactum Y at any point of Y - X.

3) For any two disjoint sets V' and V'', open in X, we have:

$$0 \langle V' \cup V'' \rangle = 0 \langle V' \rangle \cup 0 \langle V'' \rangle.$$

4) If the set F, closed in X, splits the space X into the sets U_1 and U_2, then $cl_Y(F)$ splits the compactum Y into the sets $0 \langle U_1 \rangle$ and $0 \langle U_2 \rangle$.

Proof. 1) $\Rightarrow$ 2): Let Y be a perfect compactification of X. Assuming the annex $Y - X$ splits the compactum at some point $\xi \in Y - X$, we have a neighborhood U of the point ξ such that $V = U \cap X = V' \cup V''$, where V' and V'' are disjoint open sets in X, with the additional property that $\xi \in cl_Y(V') \cap cl_Y(V'')$. Since $cl_X(V') \cap cl_X(V'') \cap V = \emptyset$, it follows that

(1) $$Fr_X V' \subset Fr_X V \subset Fr_Y U.$$

Since Y is compact, we can find a neighborhood W of ξ with $cl_Y(W) \subset U$; and we let $A = W \cap V'$. So $A \subset W$, and $Fr_X V' \subset Fr_Y U$, while $cl_Y(W) \cap cl_Y(Fr_Y U) = \emptyset$; from which it follows that

(2) $$\delta(A, Fr_X V') = 1.$$

Since A is contained in the open set V', and Y is a perfect compactification, they by 5.2,

(3) $$\delta(A, X - V') = 1.$$

But $\xi \in cl_Y(A)$ and $\xi \in cl_Y(V'') \subset cl_Y(X - V')$, so we have $\xi \in [cl_Y(A) \cap cl_Y(X - V')]$; which contradicts (3).

2) $\Rightarrow$ 3): Suppose 2) holds, and yet for some pair of disjoint open sets $V', V'' \subset X$, there is a point $\xi \in [0 \langle V' \cup V'' \rangle - (0 \langle V' \rangle \cup 0 \langle V'' \rangle)]$. If

$\xi \notin cl_Y(V')$, then there is a neighborhood W of ξ such that

(4) $$W \subset 0 \langle V' \cup V'' \rangle,$$

and

(5) $$W \cap V' = \emptyset.$$

From (4), it follows that

(6) $$W \cap X \subset 0 \langle V' \cup V'' \rangle \cap X = V' \cup V''.$$

Now (5) and (6) give us:

(7) $$W \cap X \subset V''.$$

From (7), it follows that $\xi \in 0 \langle V'' \rangle$, contradicting the choice of ξ. Hence, we must have $\xi \in cl_Y(V')$. In exactly the same way, $\xi \in cl_Y(V'')$. Thus, the annex Y - X splits the compactum Y at the point ξ, contrary to the hypothesis.

3) $\Rightarrow$ 4): Let F be a closed set which splits the space X into the sets U_1 and U_2. From 3) it follows that

(8) $$0 \langle U_1 \cup U_2 \rangle = 0 \langle U_1 \rangle \cup 0 \langle U_2 \rangle.$$

But $F = X - (U_1 \cup U_2)$, so we have:

(9) $$Y - cl_Y(F) = Y - cl_Y(X - [U_1 \cup U_2]) = 0 \langle U_1 \cup U_2 \rangle.$$

Combining (8) and (9), we have:

(10) $$Y - cl_Y(F) = 0 \langle U_1 \rangle \cup 0 \langle U_2 \rangle.$$

Since $U_1 \cap U_2 = \emptyset$, then

(11) $$0\ \langle U_1\rangle \cap 0\ \langle U_2\rangle = \emptyset.$$

(10) and (11) verify that $cl_Y(F)$ splits the compactum Y into the sets $0\ \langle U_1\rangle$ and $0\ \langle U_2\rangle$.

4) $\Rightarrow$ 1): Let U be an open set in X. As the proof of lemma 5.2 shows, we always have:

(12) $$cl_Y(Fr_X U) \subset Fr_Y 0\ \langle U\rangle.$$

We prove the reverse inclusion. Let $V = X - cl_X U$. Then $Fr_X U$ splits X into the sets U and V; so that by 4), $cl_Y(Fr_X U)$ splits Y into the sets $0\ \langle U\rangle$ and $0\ \langle V\rangle$. Since no point of $Fr_Y 0\langle U\rangle$ is in $0\ \langle U\rangle$, then

(13) $$Fr_Y 0\ \langle U\rangle \subset cl_Y(Fr_X U).$$

Thus, Y is perfect with respect to the set $U \subset X$; and since U was an arbitrary open set in X, then Y is a perfect compactification of X. Thus we have proven all parts of 5.7 equivalent.

Note that all of the characterizations of a perfect compactification have to do with the splitting of the compactum (compact and Hausdorff). In fact, corollary 5.4 shows that the Stone-Čech compactification resists splitting so well as to be a perfect compactification. Hopefully we can backtrack from the Stone-Čech compactification, to find some other compactifications which are perfect. As it turns out, we can do even better than that. Our next theorem tells us how to backtrack from the Stone-Čech compactification, and even tells us how far back we can go.

5.8 Theorem. The compactification Y of the completely regular space X is perfect if and only if the natural surjection of the Stone-Čech compactification βX onto the compactification Y is a monotone map.

Proof. To prove the theorem, we prove two lemmas, which properly contain the theorem.

5.9 Lemma. Let Y be a perfect compactification of X. If Z is any compactification such that $Z \geq Y$, then the natural surjection $\varphi: Z \to Y$ is monotone.

Proof. On the assumption that φ is not monotone, we have a $\xi \in Y$ such that

$$\varphi^{-1}(\xi) = F_1 \cup F_2, \tag{1}$$

where F_1 and F_2 are disjoint closed subsets of Z. Since Z is compact, there are disjoint open sets U_1 and U_2 in Z with $F_1 \subset U_1$ and $F_2 \subset U_2$. We let $V_1 = X \cap U_1$, and $V_2 = X \cap U_2$. Since $F_1 \subset cl_Z(V_1)$, then $\xi \in cl_Y(V_1)$. Similarly, $\xi \in cl_Y(V_2)$. We denote by U, the following open set in Y:

$$Y - \varphi\,[Z - (U_1 \cup U_2)];$$

so that $\xi \in U$ and $U \cap X = V_1 \cup V_2$. Since $V_1 \cap V_2 = \emptyset$ and $\xi \in [cl_Y(V_1) \cap cl_Y(V_2)]$, then Y - X splits the compactum Y at the point ξ, contrary to the fact that the compactification Y is perfect. The lemma is proved.

5.10 Lemma. Let Z be a perfect compactification of X, and Y be a compactification of X with the properties: i) $Z \geq Y$, and ii) the natural surjection $\varphi: Z \to Y$ is monotone. Then the compactification Y is perfect.

Proof. We shall presuppose that the compactification Y is not perfect. Then there is a point $\xi \in Y - X$, at which the annex Y - X splits the compactum Y.

Let U be a neighborhood of the point ξ in Y such that:

(1) $$U \cap X = V' \cup V'',$$

where V' and V'' are disjoint open sets in X, with:

(2) $$\xi \in [cl_Y(V') \cap cl_Y(V'')].$$

Since $\xi \in 0_Y\langle V' \cup V''\rangle$, it follows that

(3) $$\varphi^{-1}(\xi) \subset \varphi^{-1}[0_Y\langle V' \cup V''\rangle] \subset 0_Z \langle V' \cup V''\rangle = 0_Z\langle V'\rangle \cup 0_Z\langle V''\rangle,$$

where the later equality follows from 5.7 3).

If we assume $\varphi^{-1}(\xi)$ is contained in $0_Z\langle V'\rangle$, then

$$[\varphi^{-1}(\xi) \cap cl_Z(V'')] \subset [\varphi^{-1}(\xi) \cap cl_Z(0_Z \langle V''\rangle)] = \emptyset,$$

so that

$$\begin{aligned} [\{\xi\} \cap cl_Y(V'')] &= \{\xi\} \cap cl_Y(\varphi[V'']) \\ &= \varphi\, \varphi^{-1}[\{\xi\} \cap cl_Y(\varphi[V'']) \\ &= \varphi\, [\varphi^{-1}(\xi) \cap \varphi^{-1}\, \varphi\, [cl_Y(V'')]] \\ &= \varphi\, [\varphi^{-1}(\xi) \cap cl_Y(V'')] \\ &= \emptyset; \end{aligned}$$

that is, $\xi \notin cl_Y(V'')$, which is contrary to (2). Hence we have:

(4) $$\varphi^{-1}(\xi) \cap 0_Z\langle V''\rangle \neq \emptyset,$$

and similarly we must have:

(5) $$\varphi^{-1}(\xi) \cap 0_Z\langle V'\rangle \neq \emptyset.$$

Since $0_Z\langle V'\rangle$ and $0_Z\langle V''\rangle$ are disjoint open sets in Z, it follows that $\varphi^{-1}(\xi)$ is disconnected. Proof of 5.10 is thus complete.

Lemmas 5.9 and 5.10 do more than prove theorem 5.8. After recalling a definition, we shall prove a corollary of these lemmas which will give us information enabling us to determine when there is a minimal element in the partially ordered set of perfect compactifications of the space X.

5.11 Definition. The space N is called punctiform if every connected, compact subset of N consists of one point.

5.12 Corollary. A perfect compactification Y of the space X, is a minimal element in the partially ordered set of perfect compactifications of X if and only if Y - X is a punctiform space.

Proof. Suppose that Y is a perfect compactification of X with Y - X puntiform. For any perfect compactification of X such that $Z \leq Y$, we have a continuous surjection $f: Y \to Z$ such that $f|_X = 1_X$. It follows from 5.9 that f is monotone, so that Y - X being punctiform implies f is 1-1. It follows from ([1],XI, thm. 2.1(2)) that f is a homeomorphism of Y onto Z. Hence $f^{-1}: Z \to Y$ is, in particular, a continuous surjection, which shows $Y \leq Z$. Thus Y is a minimal element in the set of all perfect compactifications of X.

Conversely, suppose that Y is a minimal perfect compactification of X. If Y - X is not punctiform, let C be a non-degenerate continum in Y - X. Let Z be the quotient space of Y with C identified to a point [C], and let $q: Y \to Z$ be the quotient (identification) map. Then Z is a compactification of X, and q is obviously monotone. It follows from lemma 5.10 that Z is a perfect compactification of X with $Z \leq Y$. By the minimality of Y, we also have $Y \leq Z$; so that there is a continuous surjection $p: Z \to Y$ such that $p|_X = 1_X$. It follows from theorem 1.9 that $q: Y \to Z$ is a homeomorphism. The fact that $q^{-1}([C]) = C$ is not a single point contradicts q being 1-1. Thus Y - X must be puntiform. The corollary is proved.

Using corollary 5.12, and some results in the theory of continuous decompositions of compact spaces, we can produce a probe that points to the spaces having a minimal perfect compactification. The probe takes the form of the following theorem.

5.13 Theorem. The space X has a minimal perfect compactification if and only it has at least one compactification with a punctiform annex. In this case, the minimal perfect compactification Y is unique; it has a punctiform annex, and is the greatest of all compactifications with a punctiform annex.

Proof. The necessity of the condition is contained in corollary 5.12. To establish the sufficiency, we must verify the existence of a minimal perfect compactification of X. The condition gives us a compactification Y_1 of X, with punctiform annex, $Y_1 - X$. Let φ denote the natural surjection from the Stone-Čech compactification βX of X onto the compactification Y_1. We denote by $\{F\}$ the upper semicontinuous collection of point-inverses $\varphi^{-1}(y)$, $y \in Y_1$ (see [2], thm. 3-37); and let $\{\Phi\}$ be the upper semicontinuous collection of (connected) components of point-inverses $\varphi^{-1}(y)$, $y \in Y_1$. Let Y be the topological space associated with $\{\Phi\}$. By ([2], thm. 3-40), there are continuous mappings $\psi: \beta X \to Y$ and $\omega: Y \to Y_1$ having the following properties:

i) $\varphi = \omega \circ \psi$

ii) ψ is monotone

iii) ψ is light; that is, zero-dimensional.

Moreover, it is clear that Y is a compactification of X and ψ is the natural mapping. By theorem 5.8, Y is perfect.

Next, we shall show that Y is independent of the particular choice of Y_1. To this end, we only need to show that the decomposition $\{\Phi\}$ is independent of the choice of Y_1. Let B be a continuum (compact and connected set) in the annex $\beta X - X$. Since $Y_1 - X$ is punctiform, $B \subset \varphi^{-1}(y_o)$ for some $y_o \in Y_1 - X$, and therefore, $B \subset \Phi_o$ for some $\Phi_o \in \{\Phi\}$. Thus, the elements of the decomposition $\{\Phi\}$ lying in $\beta X - X$ may be described as maximal connected compact subsets of $\beta X - X$ and so the decomposition $\{\Phi\}$ is uniquely defined.

We now show that Y is the minimal perfect compactification of X. For any perfect compactification Z of X, we have the natural mapping $\theta: \beta X \to Z$. By theorem 5.8, θ is monotone so that each $\theta^{-1}(z)$, $z \in Z$, is a connected compact subset of $\beta X - X$. From what we have just shown of $\{\Phi\}$, we now have that the upper semicontinuous decomposition $\{\theta^{-1}(z) \mid z \in Z\}$ is a refinement of $\{\Phi\}$. Consequently, there is induced a natural mapping of the compactification Z onto the compactification Y; that is, $Z \geq Y$. Hence Y is the minimal perfect compactification of X. Moreover, Y - X is punctiform now follows from corollary 5.12.

It remains to be shown that Y is the largest of all the compactifications of X with punctiform annex. Since Y has been shown to be independent of the particular choice of Y_1, it follows that any compactification of X with punctiform annex precedes Y. The theorem is now proved.

Though the theory of perfect compactifications is interesting in its own right, as is witnessed by the preceding theorem; the notion of perfect compactification has further applications.

5.14 Corollary. Let Y_1 and Y_2 be compactifications of the spaces X_1 and X_2, such that the annexes $Y_1 - X_1$ and $Y_2 - X_2$ are puncitiform, and split the compacta Y_1 and Y_2 at none of their points. Then every homeomorphism between the spaces X_1 and X_2 (if any exist) can be extended to a homeomorphism between the compacta Y_1 and Y_2.

Proof. It is sufficient to show that in the case when $X_1 = X_2$, the identity homeomorphism from X_1 onto itself is extendable to a homeomorphism between the compactifications Y_1 and Y_2. So we let $X_1 = X_2$, and $\varphi: X_1 \to X_2$ where $\varphi: X_1 \to X_2$, where $\varphi = 1_X$. The properties of the annexes of Y_1 and Y_2 show that they both coincide with the unique minimal perfect compactification of X_1, and so $Y_1 = Y_2$. Hence the identity map $\psi: Y_1 \to Y_2$, is an extension of the map φ. Corollary 5.14 is established.

Since the relationships between perfect compactifications and punctiform annexes was established via a characterization of perfect compactifications in terms of proximity spaces; then it is indeed the theory of δ-spaces which is responsible for the production of the extension property in 5.14. We further support the usefulness of δ-spaces by applying the theory to study spaces which are not compact; but which, as we shall prove, can be compactified by adding a zero-dimensional annex (in the sense of ind) the addition of which does not increase the weight of the original space (a fact which must also be proven).

5.15 Definition. A Hausdorff space X is called peripherally compact if there exists in this space a basis of open sets, each of which has a compact frontier.

5.16 Definition. If X is a Hausdorff space and $\mathcal{B}'$ is a basis for X as described in 5.15, then the basis $\mathcal{B}$ derived from $\mathcal{B}'$ by supplementing $\mathcal{B}'$, with all finite intersections, finite unions, and complementation of closures of elements of $\mathcal{B}'$; will be called a π-compact basis of X.

It is important to observe: A π-compact basis consists only of open sets with compact frontiers. The cardinality of the π-compact basis $\mathcal{B}$, derived from the basis $\mathcal{B}'$, is the same as the cardinality of the basis $\mathcal{B}'$. A π-compact basis is the key to producing a compactification with a zero-dimensional annex. We prove that a π-compact basis $\mathcal{B}$ for the space X, induces a proximity relation δ on X. The relation δ then corresponds to some compactification Y of the space X. As it turns out, the compactification Y (called the π-compactification of X) has a zero-dimensional annex. We proceed with the proof of these facts.

5.17 Lemma. A peripherally compact space is regular.

Proof. Let x be a point in the peripherally compact space X. Let U be an arbitrary neighborhood of x. Since X has a basis of sets with compact frontiers, then it may be assumed that $Fr_X U$ is compact. Since X is Hausdorff, there exist a finite number of open sets $V_1, \cdots, V_n \subset X$, whose closures do not contain the point x. We now let

$$V = U - \bigcup_{i=1}^{n} cl_X(V_i),$$

so that

$$x \in V \subset cl_X(V) \subset [cl_X(U) - \bigcup_{i=1}^{n} V_i] \subset U.$$

This completes the proof that X is a regular space.

5.18 Lemma. Let X be a peripherally compact space with π-compact basis $\mathcal{B}$. If

A is a closed subset of X contained in the element $U \in \mathcal{B}$, then there is a $W \in \mathcal{B}$ such that $A \subset W \subset cl_X(W) \subset U$.

<u>Proof</u>. Since $U \in \mathcal{B}$, then $Fr_X U$ is compact. X is regular, so there is a finite number of open sets $W_1, \cdots, W_m$, whose closures do not meet the set A, and whose union contains $Fr_X U$. If

$$W = U - \bigcup_{i=1}^{m} cl_X(W_i),$$

then

$$A \subset W \subset cl_X(W) \subset [cl_X(U) - \bigcup_{i-1}^{m} W_i] \subset U.$$

The lemma is proved.

5.19 <u>Definition</u>. If $(X, \mathcal{T})$ is a peripherally compact space with π-compact basis $\mathcal{B}$, then we define a relation δ, on the power set of X as follows: $A \not\delta B$ if and only if there is a neighborhood $U \in \mathcal{B}$, such that $cl_X(A) \subset U$, $cl_X(B) \subset X - cl_X(U)$.

The relation δ in 5.19 turns out to be a proximity relation for the set X (cf. 5.21). We recall that every proximity relation δ induces a topology which we denote by $T(\delta)$. So to decide whether the relation defined in 5.19 is the relation we want, it is necessary to check whether it induces the peripherally compact topology $\mathcal{T}$ on X.

5.20 <u>Proposition</u>. The proximity space (X, δ) is consistent with the topological space $(X, \mathcal{T})$.

<u>Proof</u>. Suppose $A \subset X$, and $x \in cl_{\mathcal{T}}(A)$. Then every neighborhood of the point x

must meet the set A, so by definition of the proximity, $x \,\delta\, A$. If $x \notin cl_{\mathcal{T}}(A)$, then there is a neighborhood V of x such that $V \cap A = \emptyset$; hence there is also an element U of the π-compact basis $\mathcal{B}$, such that U is a neighborhood of x and $U \cap A = \emptyset$. According to lemma 5.18, there is a set $W \in \mathcal{B}$, with $x \in W \subset cl_X(W) \subset U$. It follows that we also have $cl_X(A) \subset X - cl_X(W)$, so that $x \not\delta A$. 5.20 is proved.

5.21 Proposition. The relation δ, defined in 5.19, is a proximity relation.

Proof. Properties 1), 3), and 4), of definition 2.1 are clearly satisfied by the relation δ. If $(A \cup B) \not\delta C$, then it easily follows that $A \not\delta C$ and $B \not\delta C$. Let $A \not\delta C$, $B \not\delta C$; we show that $(A \cup B) \not\delta C$. There are neighborhoods $U_1, U_2 \in \mathcal{B}$, for which $cl_X(A) \subset U_1$, $cl_X(B) \subset U_2$, $cl_X(C) \subset X - cl_X(U_1)$, and $cl_X(C) \subset X - cl_X(U_2)$. Letting $U = U_1 \cup U_2$, we then have:

$$cl_X(A \cup B) \subset U, \; cl_X(C) \subset X - cl_X(U).$$

Clearly, $U \in \mathcal{B}$, so that $(A \cup B) \not\delta C$; and property 2) of definition 2.1 is satisfied. To verify property 5), we let $A, B \subset X$ such that $A \not\delta B$. Then there is a set $U \in \mathcal{B}$, for which $cl_X(A) \subset U$, $cl_X(B) \subset X - cl_X(U)$. In accordance with lemma 5.18, there is a set $V \in \mathcal{B}$, such that $cl_X(A) \subset V$ and $cl_X(V) \subset U$. By regularity, there exists a set W such that $cl_X(B) \subset W$ and $cl_X(W) \subset X - cl_X(U)$. Since $cl_X(A) \subset V$ and $cl_X(X - U) \subset X - cl_X(V)$, then we have:

$$(1) \qquad A \not\delta (X - U).$$

Since $cl_X(B) \subset W$ and $cl_X(U) \subset X - cl_X(W)$, then we have:

$$(2) \qquad B \not\delta U.$$

Since $U \cup (X - U) = X$, then (1) and (2) conclude the proof that property 5) is satisfied. Hence 5.21 is proved.

5.22 Corollary. A peripherally compact space is completely regular.

Proof. The preceeding proposition showed that every peripherally compact space is a proximity space; and since every δ-space is completely regular (3.26) the corollary is established.

Of course, we now have that every peripherally compact space has a compactification. In fact, if $\mathcal{B}$ is a π-compact basis on the space X, and δ is the proximity relation defined by means of this basis, then there is a compactification of X corresponding to the relation δ.

5.23 Definition. The compactification Y of the peripherally compact space X, which corresponds to the proximity relation δ defined by means of the π-compact basis $\mathcal{B}$, is called the π-compactification associated with the π-compact basis $\mathcal{B}$.

5.24 Theorem. The annex in every π-compactification is zero-dimensional in the sense of ind.

Proof. The proof follows from the next two lemmas.

5.25 Lemma. Let Y be a π-compactification of the space X, associated with the π-compact basis $\mathcal{B}$; and Z an arbitrary compactification of the same space following the compactification Y. Then the compactification Z is perfect with respect to all the sets of the basis $\mathcal{B}$.

Proof. Let $U \in \mathcal{B}$, and $A \subset U$ be far from $Fr_X U$ in the sense of the proximity relation δ_1 corresponding to the compactification Z. So we have $cl_X(A) \cap Fr_X U = \emptyset$, which means $cl_X(A) \subset U$. By lemma 5.18, there is a set $V \in \mathfrak{B}$, for which $cl_X(A) \subset V$ and $cl_X(V) \subset U$. But then $X - U \subset X - cl_X(V)$, so $A \not\delta (X - U)$, where δ is the proximity relation assocaited with the π-compactification Y. Since the compactification Z precedes Y, then we also have $A \not\delta_1 (X - U)$. The lemma is proved.

5.26 Lemma. Let Y be the π-compactification of the space X, associated with the π-compact basis $\mathfrak{B}$. The system of sets $\{0\langle U\rangle \mid U \in \mathfrak{B}\}$ is a basis of the compactum Y.

Proof. Let ξ be any point of the compactum Y, and 0ξ be an arbitrary neighborhood of it. We select another neighborhood $0_1\xi$ of the point ξ, such that $cl_Y(0_1\xi) \subset 0\xi$. The sets $cl_Y(0_1\xi)$ and $Y - 0\xi$ do not intersect; therefore the sets $cl_X(0_1\xi \cap X)$ and $X - 0\xi$ are far apart. Hence there is a set $U \in \mathfrak{B}$, for which $cl_X(0_1\xi \cap X) \subset U$ and $X - 0\xi \subset X - cl_X(U)$. But then $\xi \in 0_1\xi \subset 0\langle U\rangle$ and $0\langle U\rangle \subset 0\xi$. The lemma is proved.

Proof of theorem 5.24. We only need to observe that the system of sets

$$\{0\langle U\rangle \cap (Y - X) \mid U \in \mathfrak{B}\},$$

is a basis of the annex Y - X, whose elements have empty frontiers in Y - X. The later follows from:

$$Fr_{Y - X}[0\langle U\rangle \cap (Y - X)] \subset Fr_Y\, 0\langle U\rangle,$$

and

$$Fr_Y\, 0\langle U\rangle = cl_Y(Fr_X U) = Fr_X U \subset X.$$

We now have for every peripherally compact space X, a compactification with a zero-dimensional annex, namely the π-compactification. Since we can pick from every basis, a basis whose cardinality equals the weight of the space, and since extending such a basis to a π-compact basis does not alter the cardinality of the basis; then for the π-compactification of the space we have a basis with the same cardinality as the weight of the space X. We have proven the following.

5.27 Corollary. (Freudenthal-Morita) Every peripherally compact space X may be embedded in a compactum with zero-dimensional (in the sense of ind) annex; furthermore, there is a compactification with zero-dimensional annex, whose weight coincides with that of the original space.

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Acknowledgement

I wish to express my sincere gratitude to Dr. Jung for introducing me to this subject and for his valuable assistance in preparing this report. Also, I thank him for preparing me to understand and possibly solve problems in related areas.

PROXIMITY SPACES WITH APPLICATIONS TO COMPACTIFICATIONS

by

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B.S. Kansas State University, 1968

AN ABSTRACT OF A REPORT

submitted in partial fulfillment of the

requirements of the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1969

ABSTRACT

A variety of questions remained unanswered in the theory of compactifications of completely regular spaces. The reason these questions are unanswered is apparently the complexity and the external nature of the available characterizations of compactifications.

In this paper we transfer problems of compactifications of completely regular spaces to corresponding problems in proximity spaces. A proximity space (δ-space) is a pair (P,δ), where P is a nonempty set and δ is a relation on the power set of P, satisfying:

1) If $A,B \subset P$, then $A \;\delta\; B$ iff $B \;\delta\; A$.
2) If $A,B,C \subset P$, then $(A \cup B) \;\delta\; C$ iff $A \;\delta\; C$ or $B \;\delta\; C$.
3) If $x,y \in P$, then $\{x\} \;\delta\; \{y\}$ iff $x = y$.
4) If $A \subset P$, then $A \not\delta \emptyset$.
5) If $A,B \subset P$ and $A \not\delta B$, then there are sets $C,D \subset P$ with $C \cup D = P$ and $A \not\delta C$, $B \not\delta D$.

We begin by proving that the class of proximity spaces coincides with the class of completely regular spaces. We then construct an isomorphism from the set $\mathfrak{B}$ of all compactifications of a given completely regular space X onto the set $\mathfrak{R}$ of all proximity spaces with 2^X as the set on which the proximity relation is defined. This isomorphism transfers problems of compactifications into problems of δ-spaces.

We next demonstrate the suitability of δ-spaces to solving compactification problems. We give a simple proof of the famous Freudenthal-Morita theorem on the existence for every peripherally compact space (also called rim compact or semi-compact) of a compactification with zero-dimensional annex and whose weight coincides with that of the original space. We also show that in the

class of perfect compactifications of a given completely regular space X, there is a minimal element if and only if there is at least one member of this class with a punctiform annex.